Heterogeneity and Aggregation in a Financial Accelerator Framework

(Preliminary Draft)

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July 31, 2006

Abstract

In this paper we present a macroeconomic model in which changes in the variance (and higher moments of the distribution) of firm's financial conditions –i.e. "distributive shocks"– are bound to play a crucial role in the determination of output fluctuations. Firms differ by degree of financial robustness, which affect (optimal) investment in a bankruptcy risk context (à la Greenwald-Stiglitz). As to households, for the sake of simplicity, we assume that they are homogeneous in every respect so that we can adopt the representative agent hypothesis.We can explore the properties of the macro-dynamic model either via the study of the two-dimensional map defining the laws of motion of the average equity ratio and of the variance of the distribution or via simulations in a multiagent framework.

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1 Introduction

The research on the role of individual characteristics in shaping micro-behaviour and on the related issue of agents' heterogeneity as a determinant of macroeconomic outcomes has been a thriving industry in the profession for the last ten years or so.[1] The representative agent assumption is still the cornerstone of most of contemporary macroeconomics but the awareness of its limitations [2]is spreading well beyond the circle of more or less dissenting economists. Also in mainstream macroeconomics, in fact, the representative agent is not as eagerly embraced as in the early years of the debate on micro-foundations in the remote '70s. A certain aura of skeptical disenchantment is surrounding the representative agent assumption which is still adopted mainly for lack of a workable alternative.

In order to take heterogeneity seriously in macroeconomic modelling, one should start with heterogeneous behavioural rules at the micro level and determine the aggregate (macroeconomic) quantity – such as GDP – by adding up the levels of a myriad of individual quantities. The increasing availability of computational power has allowed the implementation of this bottom-up procedure in multi-agent models. Not surprisingly, in the last ten years or so, a proliferation of agent-based models has paralleled the diffusion of research on issues concerning heterogeneity. [3]

Multi-agent modelling is the most straightforward way of tackling the heterogeneity issue. In the profession at large, however, there is no agreement on the opportunity of following this methodology. While some colleagues, mainly in the unorthodox camp, eagerly embrace the new research strategy, some others, mainly in the mainstream, are skeptical or even dismissal. There are at least three reasons for this skepticism: (i) a basic distrust for the output of computer simulations, which is generally very sen-

[1] The literature on issues pertaining to heterogeneity is growing at a very high exponential rate. See Hommes (2006) for a recent survey of models of agents'heterogeneity in financial markets. An overview of the role of heterogeneity in macro-dynamic models in Delli Gatti and Gallegati (200..)

[2] See Stoker (1997) for a detailed historical account of the development of the representative agent assumption and a thorough critique of its use and misuse.

[3] See Tesfatsion (2006) for a survey of agent based models.

sitive to the choice of initial conditions and parameter values; (ii) a critique of adaptive micro-behavioural rules which are often considered ad hoc; (iii) the difficulty and sometimes the impossibility of thinking in macroeconomic terms, i.e. of using macro-variables in the theoretical framework.

The first type of skepticism is rapidly fading away. After all, also Real Business Cycle models are too complicated to be solved by pen and paper and must be simulated. In order to do so RBC theorists have developed procedures to calibrate their models which, with the passing of time and the spreading in the profession, have become standard tools – we can even call them protocols – of macroeconomic research. [4]The same is true of agent-based models: Calibration and validation is ranking high in the agenda of multi–agent models' implementation.

As to the behavioural rules at the micro-level, it is true that some of the most enthusiastic believers in the heterogeneity mantra have seized the opportunity of agent based modelling to implement complex adaptive systems. Multi-agent models, in fact, allowe the comparison of the impact of different behavioural rules of thumb, which are often traced back to bounded rationality and adaptive behaviour. There is no reason, however, to assume that this is the only way of modelling individual choices. The multi-agent framework can also accomodate models of optimizing behaviour of heterogeneous agents. The model presented in this paper is a case in point as we will argue in a while.

Finally, the difficulty of thinking in macroeconomic terms can be circumvented by means of an appropriate aggregation procedure. In this paper we adopt a stochastic aggregation procedure – labelled the Variant-Representative Agent with a somewhat paradoxical touch – which allows to resume macroeconomic thinking in a multi-agent framework. [5] We claim that this aggregation procedure is a feasible alternative to the Representative Agent.

[4]The crux of the matter with RBC in fact is the content and the prediction of the theory..but also calibration...

[5]The procedure has already been used. See Agliari et al. (2000). It is thoroughly discussed and compared with other aggregation procedures in Gallegati et al. (2006).

The ambitious and apparently contradictory aim of the paper consists in building a macrodynamic model starting from the assumption, well corroborated by the existing evidence, that firms differ from one another according to their financial conditions, captured by the *equity ratio*, that is the ratio of the equity base or net worth to the capital stock, an indicator of financial robustness. The diversity of firms' financial conditions is the only type of heterogeneity in the present framework. For the sake of analytical tractability, we keep the degree of heterogeneity at the lowest possible level, i.e. only one type of heterogeneity for only one class of agents. [6] Therefore, we will stick to the old-fashioned representative agent assumption as far as households are concerned.

2 From micro to macro and return

Starting from the distribution of the firms' equity ratio, we build a macrodynamic model in six steps. First of all we derive a behavioural rule at the microeconomic level for investment activity in an optimizing framework (section 4). We adopt an optimizing perspective precisely to show that a multi-agent framework can accomodate optimizing behavior. Following Greenwald and Stiglitz (1993) each firm is assumed to maximize expected profit less expected bankruptcy costs. From the optimization we derive individual investment as a non-linear function of the individual equity ratio given the level of some macroeconomic variables – i.e. variables which are uniform across agents – such as the real wage and the real interest rate.

Second, we apply the aggregation procedure mentioned above to the individual investment functions. Due to the non-linearity of the individual equations, we obtain average investment as a non-linear function of the moments of the distribution of the equity ratio. For the sake of simplicity, we cut short the procedure and consider only the first and second moments, that is the *mean and the variance of the equity ratio*. The distribution of the

[6] The analysis becomes rapidly more complicated if we add just another form of heterogeneity as shown in the extension of the present model to take into account two types of heterogeneity (i.e. size and financial condition) in one class of agents (firms) presented in Assenza (2006), chapter 5.

equity ratio is changing over time [7] and affects investment accordingly. In a sense, changes in the distribution of the equity ratio, proxied by changes in the first and second moments act as shocks of a distributive nature on investment.

Investment is a crucial part of any macroeconomic story. It is the most volatile component of aggregate demand and an engine of aggregate supply inasmuch as it expands capacity. Therefore, the changing distribution of financial conditions affects aggregate demand and supply through investment. The third step in developing the macroeconomic model consists in framing investment in a general equilibrium context in which aggregate demand and supply interact. In order to do so, we have to analyze households' behaviour (section 5). We keep things as simple and as close to the mainstream conceptual framework as possible: The representative household chooses the optimal consumption plan and desired money balances maximizing utility over an infinite horizon subject to a sequence of budget constraints which incorporate money and bonds.

In equilibrium on the goods market, the consumption of the representative agent, together with investment and Government expenditure, yields a relationship between the interest rate and the output gap reminiscent of a IS curve. In equilibrium on the money market, the demand for money of the representative agent, together with the (exogenous) money supply yields a relationship between the interest rate, the output gap and real money balances reminiscent of a LM curve. Finally, on the supply side, the relationship between the scale of activity economywide proxied by the output gap and investment yields a relationship between the interest rate and the output gap which is an AS curve in our context. In the end we obtain a simple macroeconomic model, which can be solved for the equilibrium values of the interest rate, the output gap and the price level (section 6). All of the endogenous variables in equilibrium turn out to be functions of the moments of the distribution of the equity ratio which affect both the IS and AS curve

[7] Greenwald and Stiglits do consider the variance of agents'net worth as an important factor in the reaction of the macroeconomy to a shock but do not take up this issue in a dynamical context.

through investment.

Steps one, two and three are the milestones of a route from the micro to the macro level. In a sense they provide the microfoundations of a macroeconomic model with heterogeneous agents. The difference between the traditional microfoundations based on the representative agent and the new ones is the explicit consideration in the latter of the moments of the distribution of the equity ratio.[8] Since moments are a concise measure of the shape of the distribution, by focusing on moments we resume macroeconomic thinking in its purest form, i.e. at a general, non microeconomic, level.

So far, we have treated the moments of the distribution as pre-determined variables. In order to endogenize the dynamics of the moments, we have to go back to the micro level and focus on the law of motion of the individual equity ratio which is a function, among other things, of the interest rate (section 7). The fourth step consists in plugging the equilibrium value of the interest rate – which is a function of the moments of the distribution – into the individual law of motion. As a consequence, the current equity ratio turns out to be a function not only of the individual lagged equity ratio but also of the lagged average equity ratio and variance. A mean field effect is at work: the average or macro state variable, in fact, affects the micro state variable. In a sense we incorporate a macrofoundation of the micro-dynamics.

The fifth step consists in describing the dynamics of the moments. Two paths can be followed. The individual law of motion can be simulated in a multi-agent setting and macroeconomic aggregates can be determined by adding up individual quantities. The moments are computed directly from the empirical distribution obtained from simulated data. As an alternative, one can apply an aggregation procedure to the individual law of motion and

[8] In our simple case, since we cut short the aggregation procedure and consider only the first and second moments, the only difference between old and new microfoundations is the variance of the distribution. In fact the first moment, i.e. the mean of the equity ratio, would be present also in the traditional microfoundations. The equity ratio of the representative agent coincides with the mean of the distribution of the equity ratio when the variance is zero.

determine a two dimensional non-linear dynamic system in discrete time which describes the evolution over time of the mean and the variance of the distribution itself.

The sixth and final step consists in exploring the feedback from the dynamics of the moments to the aggregate variables. Both the steady state solution and transitional dynamics are interesting determinants of the aggregate.

3 The environment

We consider a closed economy populated by firms, households, financial intermediaries (banks) and the public sector (Government).

Firms will be indexed by $i = 1, 2, .., z$ with z "large". They produce a homogeneous good by means of capital and labor and invest in order to expand capacity. Each firm is characterized by a certain degree of financial robustness, captured by the equity ratio that is the ratio of the equity base or net worth to the capital stock $a_{it} = \dfrac{A_{it}}{K_{it}}$.

As to households, for the sake of simplicity, we assume that they are homogeneous in every respect so that we can adopt the representative agent hypothesis. Households demand consumption goods, financial assets (bonds) and money balances (deposits) and supply labor services. The interest rate on bonds is the opportunity cost of holding money. The allocation of households' savings to bonds and money, therefore, depends on the interest rate on bonds.

Banks receive deposits from households and extend credit to firms at an interest rate which is equal to the interest rate on bonds. Assuming, for the sake of simplicity, that there is no currency, base money coincides with banks' reserves at the central bank. In this case, money coincides with deposits and is somehow controlled by the central bank. We assume that deposits are not remunerated.

The Government carries on public expenditure, which can be thought of as investment (for instance in infrastructures). For the sake of simplicity it does not raise taxes. Therefore the budget deficit coincides with Government expenditure.

In this economy there will be markets for labor, goods and financial assets. We will not impose an equilibrium condition on the labor market. In other words the labor market can be characterized by underemployment associated with equilibrium on the money and goods markets. Thanks to Walras' law, there will be also equilibrium on the market for financial assets.

4 Firms

Firms will be indexed by $i = 1, 2, .., z$ with z "large". They produce a homogeneous good by means of capital and labor and invest in order to expand capacity. The assumption of a large number of firms which produce an undifferentiated good implies that the market structure is competitive, i.e. firms are price takers.

Firms are heterogeneous with respect to their financial robustness captured by the equity ratio a_{it}. In other words, the equity ratio of the i-th firm at time t a_{it} is a random variable with support (0,1), whose distribution is characterized by expected value $E(a_{it}) = a_t$ and variance $E(a_{it} - a_t)^2 = V_t$. The expected value is the equity ratio of the average agent (average equity ratio for short). The variance measures the dispersion of the actual equity ratios around the average. The representative agent is a particular case of this framework: it coincides with the average agent when the variance is zero. In other words the representative agent is the zero-variance average agent.

Each firm is subject to an idiosynchratic shock on revenue, such as a sudden change in consumers' preferences or a technological shock. For each firm the shock is the realization of a uniformly distributed random variable u_i with support (0,2), so that $E(u_i) = 1$.

Firms cannot raise external finance on the equity market (due to equity rationing: Myers and Majluf, 1984; Greenwald *et al.,* 1984) so that they have to rely on credit to finance investment. Therefore, they run the risk of bankruptcy.The interest rate charged by the banks will be denoted with r and is uniform across firms.

Technology and market structure. Each firm carries on production by means of a constant returns to scale technology that uses labor and capital as inputs. For simplicity we assume that technology is of the Leontief type and uniform across firms.The production function of the i-th firm therefore is $Y_i = \min(\lambda N_i, \nu K_i)$ where Y_i, N_i and K_i represent output, employment and capital (in the current period, i.e. at time t), ν and λ are positive parameters which measure the productivity of capital and labour respectively.

Assuming that labour is always abundant, we can write $Y_i = \nu K_i$ and $N_i = \frac{\nu}{\lambda} K_i = \frac{Y_i}{\lambda}$. ν is the reciprocal of the capital/output ratio. $\frac{\nu}{\lambda}$ is the reciprocal of the capital/labour ratio. Since these parameters are constant, by assumption output, capital and employment grow at the same rate. We will determine the rate of capital accumulation endogenously (see below) and will assume that output and employment grow at the same rate of the capital stock.

Profit. Profit of the i-th firm in real terms in the current period (π_i) is the difference between revenues ($u_i Y_i$) and total costs, which consist of production costs ($wN_i + rK_i$) and adjustment costs $\left(\frac{1}{2}\frac{I_i^2}{\bar{K}}\right)$:

$$\pi_i = u_i Y_i - wN_i - rK_i - \frac{1}{2}\frac{I_i^2}{\bar{K}} \tag{1}$$

u_i is the average revenue of the firm. For the sake of simplicity we assume that it is a random variable uniformly distributed over the interval $(0, 2)$ with $E(u_i) = 1$ where $E(u_i)$ is the expected value of the firms' average revenues, i.e. the expected average revenue; w is the real wage rate, r is the real interest rate, $I_i = K_i - K_{it-1}$ is investment[9] and $\bar{K} = \frac{K}{z}$ is the average capital stock, $K = \sum_{i=1}^{z} K_i$ being the aggregate capital stock.For the moment we assume that the real wage is given and constant.

Adjustment costs are quadratic in investment (as usual in investment theory) and decreasing in the average capital stock, i.e. we assume a *positive externality* in the accumulation of capital: the higher the economywide capital stock, the lower adjustment costs for the single firm. This is essen-

[9] For the sake of simplicity we assume that there is no depreciation.

tially a technical assumption, which allows to determine a relatively simple interior solution to the firm's optimization problem (see below).

Bankruptcy. The probability of bankruptcy for the i-th firm depends, among other things, on the equity ratio (see the appendix for a discussion). For the sake of analytical tractability we assume that firms adopt the following proxy of the probability of bankruptcy:

$$\Phi_i \approx \frac{\alpha}{a_{it-1}} \tag{2}$$

where $0 < \alpha < 1$. From (2) follows that the firm goes bankrupt with probability one if the equity ratio falls to α. Therefore the minimum of the equity ratio, i.e. the threshold the firm should not pass otherwise it goes bankrupt is α. On the other hand, since the maximum equity ratio is one, the minimum probability of bankruptcy is α. Hence both a_{it-1} and Φ_i are defined on the interval $(\alpha, 1)$. Also the definition (2) is essentially a technical assumption. More complicated formulations of the probability of bankruptcy would have made the model very difficult to manage without adding much to the results.

Bankruptcy is costly and the cost of bankruptcy is an increasing linear function of the capital that the firm owns, i.e. $CB_i = \beta K_i$ where β is a positive parameter.

The objective function of the firm V_i is the difference between expected profit $E(\pi_i)$ and bankruptcy (or borrower's) risk, i.e. bankruptcy cost in case bankruptcy occurs $CB_i\Phi_i$:

$$V_i = E(\pi_i) - CB_i\Phi_i = Y_i - wN_i - rK_i - \frac{1}{2}\frac{I_i^2}{\bar{K}} - \beta K_i \frac{\alpha}{a_{it-1}} \tag{3}$$

In case there were no bankruptcy, i.e. $\beta = 0$,(3) would boil down to

$$E(\pi_i) = Y_i - wN_i - rK_i - \frac{1}{2}\frac{I_i^2}{\bar{K}} \tag{4}$$

i.e to expected profit. Comparing the objective functions (3) and (4) we see that (3) is smaller than (4) i.e. to expected profit.

Recall now that $Y_i = \nu K_i$ and $N_i = \dfrac{\nu}{\lambda} K_i$.As a consequence, the problem of the firm can be formulated as follows:

$$\max_{K_i} V_i = \left[\nu\left(1-\frac{w}{\lambda}\right) - r\right] K_i - \frac{1}{2}\frac{\left(K_i - K_{it-1}\right)^2}{\bar{K}} - \beta\alpha\frac{K_i}{a_{it-1}}$$

where the control variable is the individual capital stock.[10] Notice that, due to the Leontief technology, once the stock of capital has been optimally determined solving the problem above, both output and employment follow being proportional to capital.

From the FOC we obtain:

$$\tau_i = \gamma - r - \frac{\beta\alpha}{a_{it-1}} \tag{5}$$

where $\tau_i \equiv \dfrac{I_i}{\bar{K}}$ is the investment ratio and $\gamma \equiv \nu\left(1-\dfrac{w}{\lambda}\right)$

In principle τ_i can be negative. $\tau_i < 0$ if the capital stock is shrinking, i.e. $I_i = K_i - K_{it-1} < 0$, a situation which we could not exclude – due for instance to a process of "creative distruction" which requires stripping obsolete machinery – but which should be relatively rare. The most common scenario in which capital is growing occurs if $\tau_i > 0$. According to (5) $\tau_i > 0$ iff $a_{it-1} > \dfrac{\beta\alpha}{\gamma - r}$.

According to (5), ignoring technological parameters, the individual investment ratio depends on two variables which are uniform across firms (the costs of primary inputs w, r) and on one individual variable, namely the degree of financial robustness a_{it-1}. In particular, as one could expect, the investment ratio is decreasing with the costs of the primary inputs: $\tau_{iw} = -\dfrac{\nu}{\lambda} < 0, \tau_{ir} = -1 < 0$ and increasing with the equity ratio. In fact

$$\frac{\partial\tau_i}{\partial a_{it-1}} = \frac{\beta\alpha}{a_{it-1}^2} > 0$$

[10] In order to ensure that V_i is positive we impose the following restriction on parameters:

$$a_{it-1} > \beta\alpha\frac{K_i}{K_i\left[\nu\left(1-\frac{w}{\lambda}\right) - r\right] - \frac{1}{2}\frac{I_i^2}{\bar{K}}}$$

In the absence of bankruptcy costs ($\beta = 0$) we obtain the first best:

$$\hat{\tau} = \gamma - r \tag{6}$$

According to (6), in the first best the investment ratio depends only on the costs of primary inputs w, r. Of course, financial robustness a_{it-1} has no role to play. Notice that, according to intuition, in the first best the investment ratio is always greater than in the presence of the risk of default: $\tau_i = \hat{\tau} - \dfrac{\beta\alpha}{a_{it-1}}$.

The average investment ratio τ is the average of individual investment ratios:

$$\tau = \frac{1}{z}\sum_{i=1}^{z}\tau_i = \hat{\tau} - \frac{\beta\alpha}{z}\left[\frac{1}{a_{1t-1}} + \frac{1}{a_{2t-1}} + ... + \frac{1}{a_{zt-1}}\right]$$

Hence, it depends on the costs of primary inputs w and r, which are uniform across firms and on the distribution of the firms'degree of financial robustness $(a_{1t-1}, a_{2t-1}, .., a_{zt-1})$. In the following we will "summarize" the distribution with its first and second moments. In order to do so we approximate the individual investment ratio in the neighborhood of average financial robustness $(E(a_{it-1}) = a_{t-1})$ as follows:

$$\tau_i \approx \tau_R + \frac{\partial \tau_i}{\partial a_{it-1}}\mid_{a_{t-1}} (a_{it-1} - a_{t-1}) + \frac{1}{2}\frac{\partial^2 \tau_i}{\partial a_{it-1}^2}\mid_{a_{t-1}} (a_{it-1} - a_{t-1})^2$$

where:

$$\tau_R = \hat{\tau} - \frac{\beta\alpha}{a_{t-1}}$$

$$\frac{\partial \tau_i}{\partial a_{it-1}}\mid_{a_{t-1}} = \frac{\beta\alpha}{a_{t-1}^2} > 0$$

$$\frac{\partial^2 \tau_i}{\partial a_{it-1}^2}\mid_{a_{t-1}} = -\frac{2\beta\alpha}{a_{t-1}^3} < 0$$

τ_R is the investment ratio of the representative agent in the Representative Agent economy. [11]

We can compute the average investment ratio taking the expected value of the expression above:

$$\tau = E(\tau_i) \approx \tau_R + \frac{\beta\alpha}{a_{t-1}^2} E(a_{it-1} - a_{t-1}) - \frac{\beta\alpha}{a_{t-1}^3} E(a_{it-1} - a_{t-1})^2 \tag{7}$$

Notice that by definition of expected value $E(a_{it-1} - a_{t-1}) = 0$. Moreover $E(a_{it-1} - a_{t-1})^2 = V_{t-1}$ is the variance of the distribution of equity ratios. Therefore equation (7) boils down to:

$$\tau \approx \tau_R - \frac{\beta\alpha}{a_{t-1}^3} V_{t-1} = \hat{\tau} - \frac{\beta\alpha}{a_{t-1}} - \frac{\beta\alpha}{a_{t-1}^3} V_{t-1} = \gamma - r - \frac{\beta\alpha}{a_{t-1}} \left(1 + \frac{1}{a_{t-1}^2} V_{t-1}\right) \tag{8}$$

Ignoring technological coefficients, in the following we will refer to the average investment ratio with the expression:

$$\tau = \gamma - r - f(a_{t-1}, V_{t-1}) \tag{9}$$

where

$$f(a_{t-1}, V_{t-1}) = \frac{\beta\alpha}{a_{t-1}} \left(1 + \frac{1}{a_{t-1}^2} V_{t-1}\right)$$

is a function of the relevant moments of the distributions of the firms according to financial robustness [12]. In the following, we will define a *positive distributive shock* as a change in one or more of the basic features of the distributions which boosts the average investment ratio. Therefore a positive distributive shock could be an increase of a_{t-1} or a decrease of V_{t-1}.

Notice that the average investment ratio in the presence of heterogeneity τ is smaller than the investment ratio of the representative agent τ_R which in turn is smaller than the investment ratio in the first best $\hat{\tau}$.

[11] τ_R is greater than 0 iff $a_{t-1} > \frac{\beta\alpha}{\hat{\tau}}$.

[12] The approximation procedure adopts Taylor's formula. If it were more precise (i.e. if it were not interrupted at the second term) higher moments of the distribution would show up.

5 Households

We model households'behaviour in the case of Infinitely Lived agents. Moreover, in order to get rid of unnecessary complications, we adopt the representative agent assumption with respect to the household. The representative household supplies inelastically one unit of labour. Since by assumption all the profits are retained within the firm, the only source of income for the representative agent is the wage rate if employed, the unemployment subsidy if unemployed. In symbols, income is w_k with $k = u, e$ where e stands for employed and u for unemployed: $w_e = w$ (i.e. the real wage rate), $w_u = \sigma$(i.e. the unemployment subsidy).

The household saves part of its income and invests it in bonds and money balances. [13]. Money balances are desirable because they provide "liquidity services" which are necessary if transactions require a means of payment, as we will assume.

The lifetime utility function of the representative household is:

$$U_t = \sum_{s=0}^{\infty} \left(\frac{1}{1+\rho}\right)^s (c_{t+s})^{\delta} \left(\frac{m_{t+s}}{P_{t+s}}\right)^{1-\delta} \tag{10}$$

where c_t is consumption, m_t are (per capita) money balances, ρ is the rate of time preference. The felicity function $u_{t+s} = (c_{t+s})^{\delta} \left(\frac{m_{t+s}}{P_{t+s}}\right)^{1-\delta}$ is a well behaved Cobb-Douglas function with $0 < \delta < 1$.

The household's budget constraint is:

$$c_t + \frac{m_t + b_t}{P_t} = w_k + \frac{m_{t-1}}{P_{t-1}}\theta_{t-1} + (1+i)\frac{b_{t-1}}{P_{t-1}}\theta_{t-1} \tag{11}$$

where m_t, b_t are money and bonds (per capita) respectively. w_k is income, $\theta_{t-1} \equiv \frac{P_{t-1}}{P_t}$ is the real gross rate of return on money holdings. By definition, $\frac{1}{\theta_{t-1}} - 1$ is the inflation rate. The expression $(1+i)\,\theta_{t-1} = 1 + r_t$ is the real

[13] Notice that total saving consists of households'saving and firms'saving (retained profits). By assumption the retention ratio is 100%, i.e. all the profits are retained within the firm. Therefore total saving is equal to total profits plus households' savings.

gross interest rate.

According to the budget constraint, the sum of consumption and the demand for money and bonds should be equal to income plus interest payments equal to $(1+i)\, b_{t-1}$ and money balances equal to m_{t-1} (carried over from the previous period). Notice that households'saving (i.e. the change in households' wealth) is a flow variable. Therefore, we define the demand for assets as a flow variable, too.

The problem of the representative household, therefore, consists in maximizing (10) subject to (11) or:

$$\begin{aligned} &\max_{c_t, m_t/P_t, b_t/P_t} U_t = \textstyle\sum_{s=0}^{\infty} \left(\frac{1}{1+\rho}\right)^s (c_{t+s})^{\delta} \left(\frac{m_{t+s}}{P_{t+s}}\right)^{1-\delta} \\ &s.t. \quad c_t = w_k + \frac{m_{t-1}}{P_{t-1}}\theta_{t-1} + (1+i)\,\frac{b_{t-1}}{P_{t-1}}\theta_{t-1} - \frac{m_t}{P_t} - \frac{b_t}{P_t} \end{aligned} \tag{12}$$

From which we obtain the following Lagrangian:

$$\begin{aligned} L \;=\; & \sum_{s=0}^{\infty} \left(\frac{1}{1+\rho}\right)^s (c_{t+s})^{\delta} \left(\frac{m_{t+s}}{P_{t+s}}\right)^{1-\delta} + \\ & + \left(\frac{1}{1+\rho}\right)^s \lambda_{t+s} \left[w_k + \frac{m_{t-1+s}}{P_{t-1+s}}\theta_{t-1+s} + \right. \\ & \left. + (1+i)\,\frac{b_{t-1+s}}{P_{t-1+s}}\theta_{t-1+s} - \frac{m_{t+s}}{P_{t+s}} - \frac{b_{t+s}}{P_{t+s}} - c_{t+s} \right] \end{aligned}$$

The FOCs are:

$$\frac{\partial L}{\partial c_t} = \delta\,(c_t)^{\delta-1}\left(\frac{m_t}{P_t}\right)^{1-\delta} = \lambda_t \tag{13}$$

$$\frac{\partial L}{\partial m_t/P_t} = (1-\delta)\,(c_t)^{\delta}\left(\frac{m_t}{P_t}\right)^{-\delta} - \lambda_t + \left(\frac{1}{1+\rho}\right)\lambda_{t+1}\theta_t = 0 \tag{14}$$

$$\frac{\partial L}{\partial b_t/P_t} = -\lambda_t + \left(\frac{1}{1+\rho}\right)\lambda_{t+1}\,(1+i)\,\theta_t = 0 \tag{15}$$

Solving (13) (14) (15) for $c_t, m_t/P_t$ we obtain the following relation between optimal consumption and money demand:

$$\frac{m_t}{P_t} = \frac{1-\delta}{\delta}\frac{1+i}{i}c_t \tag{16}$$

We assume that changes in money supply are implemented by means of open market operations. Therefore:

$$\frac{m_t}{P_t} - \frac{m_{t-1}}{P_{t-1}}\theta_{t-1} = -\left[\frac{b_t}{P_t} - (1+i)\frac{b_{t-1}}{P_{t-1}}\theta_{t-1}\right] \tag{17}$$

Substituting (16) and (17) into (11) we obtain the optimal consumption function and money demand function for the representative household:

$$\begin{aligned} c_t &= w_k \\ \frac{m_t^d}{P_t} &= \frac{1-\delta}{\delta}\frac{1+i}{i}w_k \end{aligned}$$

Total consumption is the sum of consumption of the employed and consumption of the unemployed people. Each type of consumption, in turn, is the product of per-capita consumption times the number of agents in each group (employed and unemployed people respectively). Therefore:

$$C_t = wN_t + \sigma(L - N_t) \tag{18}$$

where L is total labour force.

6 The macroeconomic equilibrium

The goods market is in equilibrium (planned expenditure is equal to actual expenditure) if $C_t+I_t+G_t = Y_t$ where C_t is aggregate consumption, I_t is aggregate investment, G_t is government expenditure. C_t is defined as in (18). Investment is $I_t = \tau K_t$ where $\tau = \gamma - r - f$ and $f = f(a_{t-1}, V_{t-1})$(see equation (9)). As to Government expenditure we assume that it is proportional to the investment gap, i.e. the difference between the first best investment ratio and the current average investment ratio:

$$G_t = \varepsilon \left(\tau_R - \tau \right) K_t = \varepsilon f K_t \tag{19}$$

with $0 < \varepsilon < 1$. Therefore, in equilibrium the following must hold true:

$$wN_t + \sigma(L - N_t) + \left(\gamma - r - f \right) K_t + \varepsilon f K_t = Y_t \tag{20}$$

Dividing by N_t, and recalling that $\dfrac{Y_t}{N_t} = \lambda$, $\dfrac{K_t}{N_t} = \dfrac{\lambda}{\nu}$ we can rewrite (20) as follows:

$$w + \sigma \left(\frac{1}{x_t} - 1 \right) + \left(\gamma - r - f \right) \frac{\lambda}{\nu} + \varepsilon f \frac{\lambda}{\nu} = \lambda \tag{21}$$

where $x_t = \dfrac{N_t}{L}$ is the ratio of employment to population. [14]Notice that, thanks to the linearity of technology, x_t can be thought of also as the output gap. [15] Equation (21) can be solved for r, yielding

$$r = \frac{\nu}{\lambda} \left(\frac{1}{x_t} - 1 \right) \sigma - \left(1 - \varepsilon \right) f \tag{22}$$

This relation between r and x represents the IS curve of our model.

We now turn to the *money market*. Total demand for money is the sum of the demand for money of the employed and of the unemployed people. Each type of demand for money, in turn, is the product of per-capita demand times the number of agents in each group (employed and unemployed people respectively). Therefore:

$$\frac{M_t^d}{P_t} = \frac{1 - \delta}{\delta} \frac{1 + i}{i} \left[wN_t + \sigma(L - N_t) \right]$$

Recalling that the nominal interest rate is defined as $i = \dfrac{1 + r}{\theta_t} - 1$, imposing the equilibrium condition $M_t^d = \bar{M}_t$ where $\bar{M}_t$ is money supply and dividing by N_t we get

$$\frac{\bar{m}_t^s}{P_t} = \delta' \frac{1 + r}{1 + r - \theta_t} \left[w + \sigma \left(\frac{1}{x_t} - 1 \right) \right] \tag{23}$$

[14]As a consequence, the unemployment rate is $u_t = 1 - x_t$

[15]In fact, $Y_i = \nu K_i = \lambda N_i$. Hence $N_t = Y_i / \lambda$ and $L = \hat{Y} / \lambda$ where $\hat{Y}$ is potential output.

where $\delta' = \dfrac{1-\delta}{\delta}$. This relation between r and x represents the LM curve of our model.

6.1 The short run

Assume that $\theta_t = 1$ (short run price rigidity). In the short run, the macroeconomic model in structural form, consists of equations (22) (23). The reduced form consists of equations for r and x which depend on m/P, w sigma and technological parameters.

6.2 The long run

Let's now turn to the supply side of the model. Aggregate output and employment grow at the same rate as the capital stock. Therefore $x_t = (1+g)x_{t-1}$. But $1+g = \dfrac{1}{1-\tau}$. Therefore the output gap determined on the supply side of the economy will be

$$x_t = \frac{1}{1-\gamma+r+f} x_{t-1} \tag{24}$$

The macroeconomic model in structural form, therefore consists of equations (22) (23) and (24). Let's consider the steady state, i.e. $x_t = x_{t-1}$ and $\theta_t = 1$. Hence the model consists of the following equations

$$\begin{aligned} r &= \frac{\nu}{\lambda}\left(\frac{1}{x_t} - 1\right)\sigma - (1-\varepsilon) f \\ \frac{\bar{m}^s_t}{P_t} &= \delta' \frac{1+r}{r}\left[w + \sigma\left(\frac{1}{x_t} - 1\right)\right] \\ r &= \gamma - f \end{aligned}$$

From the third equation we obtain the equilibrium real interest rate:

$$r^* = \nu\left(1 - \frac{w}{\lambda}\right) - f \tag{25}$$

Note that $r^* > 0$ iff $f < \nu\left(1 - \dfrac{w}{\lambda}\right)$. Substituting this value into the IS curve we are able to get the steady state value of the output gap x^*:

$$x^* = \frac{1}{\left[\nu\left(1-\frac{w}{\lambda}\right)-\varepsilon f\right]\frac{\lambda}{\nu\sigma}+1} \tag{26}$$

Finally from the LM curve we get the steady state value of the real money balances:

$$\left(\frac{m}{P}\right)^* = \delta'\frac{1+r^*}{r^*}\left[w+\sigma\left(\frac{1}{x^*}-1\right)\right] \tag{27}$$

Changes in money supply do not affect real variables.

7 Dynamics

In this section we explore the dynamics stemming from the macroeconomic model presented in the previous section. First of all we have to establish the law of motion of the individual net worth. Assuming that there are no dividends, the level of net worth in real terms for the i-th firm in t is $A_{it} = A_{it-1} + \pi_i$. Hence

$$A_{it} = A_{it-1} + u_i Y_i - wN_i - rK_i - \frac{1}{2}\frac{I_i^2}{\bar{K}}$$

Dividing by the capital stock we obtain the law of motion of the equity ratio:

$$a_{it} = a_{it-1}\frac{K_{it-1}}{K_i} + u_i\nu - w\frac{\nu}{\lambda} - r - \frac{1}{2}\frac{I_i^2}{K_i\bar{K}} \tag{28}$$

Recall that $\frac{K_{it-1}}{K_i} = 1 - \frac{I_i}{K_i} = 1 - \frac{\bar{K}}{K_i}\tau_i = 1 - \frac{\tau_i}{k_i}$ where $k_i \equiv \frac{K_i}{\bar{K}}$. Moreover $\frac{I_i^2}{K_i\bar{K}} = \frac{I_i^2}{K_i}\frac{\bar{K}}{\bar{K}^2} = \frac{\tau_i^2}{k_i}$. Plugging these expressions into (28) we obtain:

$$a_{it} = a_{it-1}\left(1-\frac{\tau_i}{k_i}\right) + u_i\nu - w\frac{\nu}{\lambda} - r - \frac{1}{2}\frac{\tau_i^2}{k_i} \tag{29}$$

but, according to (5), $\tau_i = \nu\left(1-\frac{w}{\lambda}\right) - r - \frac{\beta\alpha}{a_{it-1}}$ and $r = \nu\left(1-\frac{w}{\lambda}\right) - f$.

Therefore:

$$\tau_i = f - \frac{\beta\alpha}{a_{it-1}} \tag{30}$$

Substituting (30) into (29), assuming that $k_i \simeq 1$ after rearranging and simplifying we get:

$$a_{it} = a_{it-1}\left[1 - f\right] + \Gamma + f - \frac{1}{2}\left[f - \frac{\beta\alpha}{a_{it-1}}\right]^2 \tag{31}$$

where $\Gamma \equiv \beta\alpha - (1 - u_i)\,\nu$. The expression above represents the individual law of motion of the equity ratio.It is a non linear first order difference equation in the state variable a_{it}.

Equation (31) can be simulated. As an alternative we can describe the trajectories of the moments of the distribution as follows. First of all we must take a linear approximation of (31). We can adopt the same aggregation procedure followed to derive the average investment ratio. This means computing

$$a_{it} \approx a_R + \frac{\partial a_{it}}{\partial a_{it-1}}\mid_{a_{t-1}}(a_{it-1} - a_{t-1}) + \frac{1}{2}\frac{\partial^2 a_{it}}{\partial a_{it-1}^2}\mid_{a_{t-1}}(a_{it-1} - a_{t-1})^2$$

where

$$a_R = a_{t-1}\left[1 - f\right] + \Gamma + f - \frac{1}{2}\left[f - \frac{\beta\alpha}{a_{it-1}}\right]^2$$

We take a linear approximation of (31). If we follow the aggregation procedure in $a_{it-1} = a_{t-1}$ and $f = \dfrac{\beta\alpha}{a_{t-1}}$ (i.e. the case of the representative agent) getting:

$$a_{it} = \Gamma_{i0} + \Gamma_1 a_{it-1} + \Gamma_2 f \tag{32}$$

where $\Gamma_{i0} = \phi + \beta\alpha = 2\beta\alpha - (1 - u_i)\,\nu$; $\Gamma_1 = 1 - \dfrac{\beta\alpha}{a_{t-1}}$; $\Gamma_2 = \bar{c}\alpha$. We take the expected value of (32) we get:

$$a_t = \Gamma_0 + \Gamma_1 a_{t-1} + \Gamma_2 f\,(D) \tag{33}$$

is the aggregate law of motion.

We know that $V_t = E\left(a_{it} - a_t\right)^2 = E\left(a_{it}\right)^2 - a_t^2$ but:

$$\begin{aligned} E\left(a_{it}^2\right) &= E\left\{\left[\Gamma_0 + \Gamma_1 a_{it-1} + \Gamma_2 f\left(D\right)\right]^2\right\} = \\ &= E\left[\Gamma_0^2 + \Gamma_1^2 a_{t-1}^2 + \Gamma_2^2 f^2\left(D\right) + 2\Gamma_0\Gamma_1 a_{it-1} + \right. \\ &\quad \left. +2\Gamma_0\Gamma_2 f\left(D\right) + 2\Gamma_1\Gamma_2 f\left(D\right) a_{it-1}\right] \end{aligned}$$

after some algebra we get:

$$\begin{aligned} E\left(a_{it}^2\right) &= \Gamma_0^2 + \Gamma_2^2 f^2\left(D\right) + 2\Gamma_0\Gamma_2 f\left(D\right) + \\ &\quad +2\Gamma_1\left[\Gamma_0 + \Gamma_2 f\left(D\right)\right] a_{t-1} + \Gamma_1^2 E\left(a_{it-1}^2\right) \end{aligned}$$

from the definition of variance we know that $V_{t-1} = E\left(a_{it-1}\right)^2 - a_{t-1}^2$ from which we get $E\left(a_{it-1}\right)^2 = V_{t-1} + a_{t-1}^2$. Finally we obtain:

$$V_t = \Gamma_0^2 + \Gamma_2^2 f^2\left(D\right) + 2\Gamma_0\Gamma_2 f\left(D\right) + 2\Gamma_1\left[\Gamma_0 + \Gamma_2 f\left(D\right)\right] a_{t-1} + \Gamma_1^2\left(V_{t-1} + a_{t-1}^2\right) - a_t^2 \tag{34}$$

the equations 32 and 34 represent a two dimensional map where the state variable are the first two moments of the distribution of fimrs by size:

$$\begin{cases} a_t = \Gamma_0 + \Gamma_1 a_{t-1} + \Gamma_2 f\left(D\right) \\ V_t = \Gamma_0^2 + \Gamma_2^2 f^2\left(D\right) + 2\Gamma_0\Gamma_2 f\left(D\right) + 2\Gamma_1\left[\Gamma_0 + \Gamma_2 f\left(D\right)\right] a_{t-1} + \Gamma_1^2\left(V_{t-1} + a_{t-1}^2\right) - a_t^2 \end{cases}$$

8 Lines of further research

We will explore the dynamic properties of the model, at the theoretical level, by means of simulations of equation (32). In a bottom up approach we will determine the dynamics of the aggregate by adding up individual levels of each variable. At the empirical level, we will explore the performance of the model by means of a VAR.

A The probability of bankruptcy

The true probability of bankruptcy can be determined as follows.

Assuming that there are no dividends, the level of net worth in real terms for the i-th firm in t is $A_{it} = A_{it-1} + \pi_i$ where $\pi_i = u_i Y_i - wN_i - rK_i - \frac{1}{2}\frac{I_i^2}{\bar{K}}$represents the profit level. We define total cost as $TC_i = wN_i + rK_i + \frac{1}{2}\frac{I_i^2}{\bar{K}}$.Hence $A_{it} = A_{it-1} + u_i Y_i - TC_i$.

A firm goes bankrupt if $A_{it} < 0$, i.e. if

$$u_i < AC_i - \frac{A_{it-1}}{Y_i} \equiv \bar{u}_i$$

where $AC_i = TC_i/Y_i$ is average cost. In words: the firm goes bankrupt if the realization of the random shock is smaller than a threshold $\bar{u}_i$ which in turn depends on equity, output, and the average cost. By assumption, the shock is a uniformly distributed random variable u_i with support $(0, 2)$, so that the probability of bankruptcy is

$$\Pr(u_i < \bar{u}_i) = \frac{\bar{u}_i}{2} = \frac{1}{2}\left(AC_i - \frac{A_{it-1}}{Y_i}\right) \tag{35}$$

Let's assume, as in the text of the paper, that the cost of bankruptcy is $CB_i = \beta K_i$. The objective function of the firm V_i is the difference between expected profit $E(\pi_i)$ and bankruptcy cost in case bankruptcy occurs $CB_i \Pr(u_i < \bar{u}_i)$:

$$V_i = E(\pi_i) - CB_i \Pr(u_i < \bar{u}_i) = Y_i - TC_i - \beta' Y_i \left(AC_i - \frac{A_{it-1}}{Y_i}\right)$$

with $\beta' = \beta/2\nu$. Rearranging one gets

$$V_i = E(\pi_i) - CB_i \Pr(u_i < \bar{u}_i) = Y_i - \left(1 + \beta'\right) TC_i + \beta' A_{it-1} \tag{36}$$

The present formalization of the probability of bankruptcy makes clear that taking into account the expected bankruptcy cost in the objective function is tantamount to incurring an extra cost equal to $\beta' TC_i$ and gaining an extrarevenue equal to $\beta' A_{it-1}$.

The formalization, however, has a clear disadvantage in terms of tractability. In fact, plugging $Y_i = \nu K_i$ and $N_i = \frac{\nu}{\lambda} K_i$ into (35) and rearranging, the probability of bankruptcy turns out to be

$$\Pr(u_i < \bar{u}_i) = \frac{\bar{u}_i}{2} = \frac{1}{2}\left\{\frac{w}{\lambda} + \frac{r}{\nu} + \frac{1}{2}\frac{\left(K_i - K_{it-1}\right)^2}{\nu \bar{K} K_i} - \frac{a_{it-1}}{\nu K_i} K_{it-1}\right\}$$

The probability of bankruptcy is decreasing with the equity ratio but it depends on a large number of parameters and endogenous variables.

Moreover, maximizing (36) with respect to K_i yields

$$\tau_i = \frac{\nu}{2 + (\beta/\nu)} - w\frac{\nu}{\lambda} - r$$

The interior solution to the maximization of V_i therefore is smaller than the first best $\hat{\tau} = \nu - w\dfrac{\nu}{\lambda} - r$ but is uniform across firms and independent of net worth. Therefore we would miss an important part of the financial fragility strory we want to tell. In order to keep net worth into the interior solution we can experiment with different bankruptcy cost functions, such as $CB_i = \beta K_i^2$. In this case however, the interior solution becomes rapidly very messy. With an acceptable loss of generality we adopt the approximation of the text.

B Aggregation

In order to illustrate the aggregation procedure, let's assume that the microscopic (individual) variable y_i is a function of the microscopic (individual) variable x_i:

$$y_i = \phi\left(x_i\right) \tag{37}$$

Let's take a linear approximation of (37) up to the second order term around $\bar{x}_i = E\left(x_i\right) = x$ where x is the mean of the distribution of the microvariable x_i

$$y_i \approx \phi\left(x\right) + \phi_x\left(x\right)\left(x_i - x\right) + \frac{1}{2}\phi_{xx}\left(x\right)\left(x_i - x\right)^2$$

where $\phi_x(.) := \dfrac{d\phi}{\partial x_i}; \phi_{xx}(.) := \dfrac{d^2\phi}{\partial x_i^2}$. Summation and averaging yields

$$y = E(y_i) \approx \phi(x) + \phi_x(x) E(x_i - x) + \frac{1}{2}\phi_{xx}(x) E(x_i - x)^2$$

but $E(x_i - x) = 0$ and $E(x_i - x)^2 = V$ where V is the variance of the distribution of the microvariable x_i. Hence

$$y \approx \phi(x) + \frac{1}{2}\phi_{xx}(x) V$$

If $\phi(.)$ were linear, ϕ_x would be a given and constant parameter and ϕ_{xx} would be zero, so that the variance of the distribution of x_i.would not affect the mean of the distribution of y_i and therefore the aggregate $Y = zy$ (where z is the number of agents). If $\phi(.)$ were non linear and concave (convex), ϕ_{xx} would be negative (positive), so that the variance of the distribution of x_i.would affect negatively (positively) the mean of the distribution of the microscopic variable y_i (and therefore the aggregate).

Assume now that there is a mean field effect, i.e. the mean of the distribution of x_i affects the individual variable y_i

$$y_i = \phi(x_i, f(x, z))$$

where $f(x, z)$ is a function of the moments of the distribution of x_i. Take a linear approximation in Taylor's series up to the second term in $E(x_i) = x$

$$y \approx \phi(x, f(x, z)) + \frac{1}{2}\phi_{xx}(x, f(x, z)) V$$

Alternative procedure. Take a linear approximation in in Taylor's series up to the first term in $E(x_i) = x, z = 0$ i.e. $f(x, z) = f(x, 0)$

$$y_i \approx \phi(x, f(x, 0)) + \phi_x(x, f(x, 0))(x_i - x) + \phi_z(x, f(x, 0))[f(x, z) - f(x, 0)]$$

Summation and averaging

$$y \approx \phi(x, f(x, 0)) + \phi_z(x, f(x, 0)) z$$

In the case of the law of motion

$$a_{it} = a_{it-1}\left[1 - f\left(a_{t-1}, V_{t-1}\right)\right] + \Gamma + f\left(a_{t-1}, V_{t-1}\right) + \tag{38}$$

$$-\frac{1}{2}\left[f\left(a_{t-1}, V_{t-1}\right) - \frac{\beta\alpha}{a_{it-1}}\right]^2 \tag{39}$$

$$\Gamma \equiv \beta\alpha - \left(1 - u_i\right)\nu.$$

$$f = \frac{\beta\alpha}{a_{t-1}}\left(1 + \frac{1}{a_{t-1}^2}V_{t-1}\right)$$

aggregation 1

$$\begin{aligned}
a_{it} &\approx A + B + C \\
A &= a_{t-1}\left[1 - f\left(a_{t-1}, V_{t-1}\right)\right] + \Gamma + f\left(a_{t-1}, V_{t-1}\right) + \\
&\quad -\frac{1}{2}\left[f\left(a_{t-1}, V_{t-1}\right) - \frac{\beta\alpha}{a_{t-1}}\right]^2 \\
B &= \frac{\partial a_{it}}{\partial a_{it-1}}\,|_{a_{t-1}}\left(a_{it-1} - a_{t-1}\right) \\
C &= \frac{1}{2}\frac{\partial^2 a_{it}}{\partial a_{it-1}^2}\,|_{a_{t-1}}\left(a_{it-1} - a_{t-1}\right)^2
\end{aligned}$$

$$\begin{aligned}
A &= a_{t-1} - \beta\alpha\left(1 + \frac{V_{t-1}}{a_{t-1}^2}\right) + \Gamma + \frac{\beta\alpha}{a_{t-1}}\left(1 + \frac{V_{t-1}}{a_{t-1}^2}\right) - \frac{1}{2}\left(\frac{\beta\alpha}{a_{t-1}^3}V_{t-1}\right)^2 = \\
&= a_{t-1} - \beta\alpha\left(1 + \frac{1}{a_{t-1}^2}V_{t-1}\right)\left(1 - \frac{1}{a_{t-1}}\right) + \Gamma - \frac{1}{2}\left(\frac{\beta\alpha}{a_{t-1}^3}V_{t-1}\right)^2
\end{aligned}$$

$$\frac{\partial a_{it}}{\partial a_{it-1}} = 1 - f + \left[f - \frac{\beta\alpha}{a_{it-1}}\right]\frac{\beta\alpha}{a_{it-1}^2}$$

$$\frac{\partial^2 a_{it}}{\partial a_{it-1}^2} = -2f\frac{\beta\alpha}{a_{it-1}^3} + 3\frac{\left(\beta\alpha\right)^2}{a_{it-1}^4}$$

$$
\begin{aligned}
\frac{\partial^2 a_{it}}{\partial a_{it-1}^2} &\mid \quad {}_{a_{t-1}} = -2f\frac{\beta\alpha}{a_{t-1}^3} + 3\frac{(\beta\alpha)^2}{a_{t-1}^4} = \\
&= -2\frac{(\beta\alpha)^2}{a_{t-1}^4}\left(1+\frac{V_{t-1}}{a_{t-1}^2}\right) + 3\frac{(\beta\alpha)^2}{a_{t-1}^4} = \\
&= \frac{(\beta\alpha)^2}{a_{t-1}^4} - 2\frac{(\beta\alpha)^2}{a_{t-1}^6}V_{t-1}
\end{aligned}
$$

$$
\begin{aligned}
a_t &\approx a_{t-1} - \beta\alpha\left(1+\frac{1}{a_{t-1}^2}V_{t-1}\right)\left(1-\frac{1}{a_{t-1}}\right) + \Gamma - \frac{1}{2}\frac{(\beta\alpha)^2}{a_{t-1}^6}V_{t-1}^2 + \\
&\quad + \frac{1}{2}\frac{(\beta\alpha)^2}{a_{t-1}^4}V_{t-1} - \frac{(\beta\alpha)^2}{a_{t-1}^6}V_{t-1}^2 \\
&= a_{t-1} - \beta\alpha\left(1+\frac{1}{a_{t-1}^2}V_{t-1}\right)\left(1-\frac{1}{a_{t-1}}\right) + \Gamma - \frac{1}{2}\frac{(\beta\alpha)^2}{a_{t-1}^4}V_{t-1}\left[3\frac{V_{t-1}}{a_{t-1}^2} - 1\right]
\end{aligned}
$$

aggregation 2

$$
\begin{aligned}
a_{it} &\approx A' + B' + C' \\
A' &= a_{t-1}\left[1 - f\left(a_{t-1}, 0\right)\right] + \Gamma + f\left(a_{t-1}, 0\right) - \frac{1}{2}\left[f\left(a_{t-1}, 0\right) - \frac{\beta\alpha}{a_{t-1}}\right]^2 \\
B' &= \frac{\partial a_{it}}{\partial a_{it-1}} \mid_{a_{t-1}, f0} \left(a_{it-1} - a_{t-1}\right) \\
C' &= \frac{\partial a_{it}}{\partial f} \mid_{a_{t-1}, f0} \left[f - f\left(a_{t-1}, 0\right)\right]
\end{aligned}
$$

con $f\left(a_{t-1}, 0\right) = \dfrac{\beta\alpha}{a_{t-1}}$

$$
\begin{aligned}
A' &= a_{t-1}\left(1-\frac{\beta\alpha}{a_{t-1}}\right) + \Gamma + \frac{\beta\alpha}{a_{t-1}} - \frac{1}{2}\left[\frac{\beta\alpha}{a_{t-1}} - \frac{\beta\alpha}{a_{t-1}}\right]^2 = \\
&= a_{t-1} - \beta\alpha + \Gamma + \frac{\beta\alpha}{a_{t-1}} = a_{t-1} - \beta\alpha\left(1-\frac{1}{a_{t-1}}\right) + \Gamma
\end{aligned}
$$

$$
\frac{\partial a_{it}}{\partial a_{it-1}} \mid_{a_{t-1}, f0} = 1 - \frac{\beta\alpha}{a_{t-1}}
$$

$$\frac{\partial a_{it}}{\partial f}\ |_{a_{t-1},f0}= 1 - a_{t-1}$$

$$\begin{aligned} a_t &\approx a_{t-1} - \beta\alpha\left(1 - \frac{1}{a_{t-1}}\right) + \Gamma + (1 - a_{t-1})\frac{\beta\alpha}{a_{t-1}^3}V_{t-1} = \\ &= a_{t-1} + \beta\alpha\left[\left(\frac{1}{a_{t-1}} - 1\right)\left(1 + \frac{V_{t-1}}{a_{t-1}^2}\right)\right] + \Gamma \end{aligned}$$

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