Optimal Monetary Policy Rules: The Problem of Stability Under Heterogeneous Learning

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Abstract

In this paper we extend the analysis of optimal monetary policy rules in terms of stability of the economy, started by Evans and Honkapohja (2003b), to the case of heterogeneous learning. Following Giannitsarou (2003), we pose the question about applicability of the representative agent hypothesis to learning. This hypothesis was widely used in learning literature at early stages to demonstrate convergence of an economic system under adaptive learning of agents to one of the rational expectations equilibria in the economy. Studying the validity of this hypothesis in application to monetary policy rules is motivated by relatively recent appearance of general conditions of stability under heterogeneous learning in the literature (see Giannitsarou (2003) and Honkapohja and Mitra (2006)), indications that these conditions do not always coincide with stability conditions under homogeneous learning, and the fact that, to our knowledge, optimal monetary policy rules were not studied methodologically for stability of the system under heterogeneous learning.

We test these monetary policy rules in the general setup of New Keynesian model that is a working horse of monetary policy models today. It is of interest to see that the results obtained by Evans and Honkapohja (2003b) for homogeneous learning case are replicated for the case when the representative agent hypothesis is lifted.

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1 Introduction

Stabilization monetary policy design problem is very often studied in the New Keynesian model. Using the environment of this model, we may study different monetary policy rules to find out which is more efficient in smoothing business cycles fluctuations and also which monetary policy rule would not lead to indeterminacy of equilibria in our model. For a comprehensive overview of various interest rate rules in the New Keynesian model, one can address Woodford (2003). Also, a very often cited work on monetary policy design is by Clarida, Gali and Gertler (1999, 2002). Svensson (1999) gives a clear distinction between instrument and target rules and implications of their use.

There has been analysis of the Taylor–type rules for indeterminacy. See, e.g. Ben-Habib, Schmitt-Grohe and Uribe (2000) and Bullard and Mitra (2002). Example of indeterminacy analysis of an optimization-based policy rule in the New Keynesian model with learning is also given in Evans and Honkapohja (2003b).

A number of recent studies also consider the New Keynesian model environment with adaptive learning of agents. Examples are works of Evans and Honkapohja (2003a, 2003b) and Honkapohja and Mitra (2006) on stability of the model economy under various policy rules. Evans and Honkapohja (2003a, 2003b) take up the issue of stability under learning for optimal monetary policies (in economies with adaptive learning).

It makes sense to study for efficiency such policy under which it is clear where our economy ends up. Especially this is important in models where indeterminacy is possible. We first need to try to find out where our economy may end up under a given policy. Here, adaptive learning may serve as a selection device. Even if our rational expectations (RE) model has a unique equilibrium, it is still of interest to see if the RE hypothesis holds under learning, which is done by checking if our model under learning converges to a given rational expectations equilibrium (REE). In both cases, we will have to check certain stability conditions. After the study of stability conditions, the next step would be to study policy rules for effectiveness

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and indeterminacy, assuming or making sure that the stability conditions on the model structure are satisfied.

That is why, before we start with studying particular monetary policies for efficiency (evaluating a particular type of policy: Taylor–rule, optimization-based rule with or without commitment), we should take a general type of a linear policy feedback rule, plug it into our structural form of the New Keynesian model and obtain some general linear reduced form (RF) of this model. All things being equal (the same structural equations: Phillips and IS curves), we can have different RFs depending on the policy rule used by the policy-maker. Hence, we obtain different REEs and different stability results. Then we should study a given reduced form for stability in order to see if a given REE equilibrium will be chosen. In this paper, we will study stability of a New Keynesian model economy under the following classification of policy rules introduced by Evans and Honkapohja (2003b).

Depending on the assumptions of the central bank about the expectations of the private agents (firms, households), they divide all policy rules into fundamentals based rules and expectations based rules. Fundamentals based rule is obtained if the policy–maker assumes RE of private agents, while the expectations based rule takes into account possibly non–rational expectations of agents (assuming that these expectations are observable to the central bank).[1]

We consider stability question under assumption of heterogeneous learning of agents. As has been shown in Giannitsarou (2003) and Honkapohja and Mitra (2006), stability results may be different under homogeneous and heterogeneous learning. Honkapohja and Mitra (2006) also demonstrate that stability may depend on the interaction of structural heterogeneity and learning heterogeneity, and Honkapohja and Mitra (2005) examine how structural heterogeneity in the New Keynesian model may affect stability results under various types of policy rules.

It turns out that under fundamentals–based linear feedback policy rule (optimization–based), learning in our model will never converge to the REE of the model. Evans

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[1] We should note here that in Taylor–type rules the current value of interest rate depends on the current values of inflation and output gap. In this paper we study stability under feedback rules that are derived from the minimization problem, in particular, study their two categories, according to Evans and Honkapohja (2003): fundamentals based and expectations based. Stability under Taylor–type rules, which do not fall under this classification, will be studied later in a separate work.

and Honkapohja (2003b) demonstrate this instability result for homogeneous recursive least squares (RLS) or stochastic gradient (SG) learning[2], while we obtain a similar instability result for the three types of heterogeneous learning considered by Giannitsarou (2003).

The other category of policy rules — expectations based rules — is meant to react to agents' expectations. Under certain conditions, we can have stability. Evans and Honkapohja (2003b) obtain stability result for homogeneous RLS or SG learning. We obtain a stability result (with conditions on the model structure) for the case of the three types of heterogeneous learning.

Originally, when heterogeneous learning in a general setup of structurally homogeneous stochastic models with adaptive learning was studied by Giannitsarou (2003), the purpose of introducing heterogeneous learning of agents was to see if the representative agents hypothesis influences stability results, i.e. if we can always apply this hypothesis. For some cases, it is demonstrated, it does make sense to consider heterogeneous setup. Our paper is about stability under monetary policy rules, so, though we, in fact, prove that the representative agent hypothesis holds true for the New Keynesian model, the accent of our paper is shifted away from testing the importance (influence) of the representative agent hypothesis.

We, essentially, apply stability analysis of the model under heterogeneous learning in the same manner as stability analysis of the model under homogeneous learning is applied by Evans and Honkapohja (2003b). They study stability conditions under monetary policy rules for the case of homogeneous RLS or SG learning. Their major input is (both for one–sided learning and two–sided learning) to have shown that under fundamentals based rules the REE of the model is always unstable, while under the expectations based rule there always stability. In the two cases the reduced form of the model is different, which has as a consequence the difference in the stability results. So, the policy implication of such a stability analysis is that,

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[2]Honkapohja and Mitra (2006) and we in this paper consider two possible algorithms used to reflect bounded rationality of agents: RLS and SG learning algorithms (which are examples of econometric learning). Their description can be found, e.g., in Evans and Honkapohja (2001), Honkapohja and Mitra (2006), Giannitsarou (2003), Evans, Honkapohja and Williams (2005). Both are used by agents to update the estimates of the model parameters. Essentially, the difference is as follows. RLS algorithm has two updating equation: one—for updating parameters entering the forecast functions, the other—for updating the second moments matrix (of the model state variables). SG algorithm assumes this matrix fixed.

given the structure of the model (the two structural New Keynesian equations), the central bank can influence (determine) the outcome of its policy by selecting the appropriate monetary policy: the one that guarantees convergence to a particular REE. If there are multiple RE equilibria, the policy in some cases may be selected in such a way that would guarantee convergence to the most preferred equilibrium (from the point of view of the stabilization policy).

We, in fact, link the study of stability conditions under a certain category of linear monetary policy rules of Evans and Honkapohja (2003b) with the study of stability under heterogeneous learning of Giannitsarou(2003). We first show that in the New Keynesian–type of models, stability can be analyzed using the structural parameters, whatever the type of heterogeneous learning used using the general criterion of Honkapohja and Mitra (2006). These results are structural matrix eigenvalues sufficient, and necessary conditions for stability of structurally homogeneous model derived in this paper and aggregated economy sufficient conditions derived in Bogomolova and Kolyuzhnov (2006), where the concept of stability under heterogeneous learning, known as δ–stability, is introduced. Then we apply these results to derive stability and instability results under heterogeneous learning for the two categories of feedback rules: fundamentals based and expectations based, in the model with arbitrary number of agent types.

Summarizing all the above, our work now looks, on the one hand, like a link between the study of stability under monetary policy rules for homogeneous learning of Evans and Honkapohja (2003b) and the study of stability conditions under heterogeneous learning of Giannitsarou (2003), the link – through the δ–stability conditions derived by us for the general setup of Honkapohja and Mitra (2006) and through the general stability criterion of Honkapohja and Mitra (2006). On the other hand, this study can serve as one more economic example demonstrating application of δ–stability sufficient conditions.

The structure of the paper is as follows. In the next section we present the basic New Keynesian model. In Section 3 we discuss general stability results under heterogeneous learning. In Section 4 we give necessary, and sufficient conditions for δ–stability for structurally homogeneous models. Section 5 describes two types of optimal policy rules and structure of reduced forms under each type. In Section 6

we provide stability and instability results for types of optimal monetary policies considered in application to the New Keynesian model. Section 7 concludes.

2 Model

The model that we consider is a general New Keynesian model with observable stationary AR(1) shocks. The structural form of the model looks as follows

$$x_t = c_1 - \phi \left(i_t - \widehat{E}_t \pi_{t+1} \right) + \widehat{E}_t x_{t+1} + \chi_1' w_t$$
$$\pi_t = c_2 + \lambda x_t + \beta \widehat{E}_t \pi_{t+1} + \chi_2' w_t,$$

where the first equation is for the IS curve and the second equation is for the Phillips curve.

$w_t = \begin{bmatrix} w_{1_t} & ... & w_{k_t} \end{bmatrix}'$ - observable AR(1) shocks[3], $w_{i_t} = \rho_i w_{i_{t-1}} + \nu_{i_t}$, $|\rho_i| < 1$, $\nu_{i_t} \sim iid\left(0, \sigma_i^2\right)$, $i = 1, ..., k$

This model is a general formulation of models derived from microfoundations that are considered in macroeconomics and monetary economic literature. The two basic equations of the New Keynesian model, which are the Phillips curve and the IS curve are derived from the optimal problems of the representative household and the representative monopolistically competitive firm, with the assumption of

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[3]Typically, New Keynesian models include only observable component (which is assumed to follow an AR(1) process). However, there are specifications including both observable and unobservable shocks. For example, Evans and Honkapohja (2003), who study stability rules under recursive least squares learning, include unobservable shocks to the New Keynesian model equations. In our case more the general specification with unobservable shocks would contain additional term $\Omega_1 \epsilon_t$ in IS curve and $\Omega_2 \epsilon_t$ in the Phillips curve, where $\epsilon_t = \begin{bmatrix} \epsilon_{1_t} & ... & \epsilon_{m_t} \end{bmatrix}'$ are unobservable shocks, $\epsilon_{i_t} \sim iid\left(0, \gamma_i^2\right)$, $i = 1, ..., m$, not correlated with observable shocks g_t. Of course, these unobservables do not bring the difference into the stability results (that is why we omit them in the model analyzed), but introducing them into the setup has its own reasoning.

Introduction of unobservable shocks to structural equations when we consider central bank learning structural coefficients of the model. If we have only observable shocks (which play a role of just another regressor – some exogenous variable) as well as other observable regressors, we will evaluate the equations' coefficients exactly if we have a sufficient number of observations. Then learning is trivial. (The convergence will be very quick if initially we did not have enough observations, but gained them over a short period of time).

If we think of how these unobservable shocks can emerge at the micro foundations level, we may think of the following economic interpretation. For example, let us assume that preference and technology shocks consist of observable and unobservable components. As for preference shocks, we can imagine a qualitative change in our preferences, so that we know how the shock has changed our preferences qualitatively, but we cannot precisely measure this change quantitatively. A similar interpretation can be given to the technological shock. What we have measured enters as observable component, while the measurement error (which always exists since we assume that our quantitative measurement of any change is imprecise) is treated as unobservable component.

Calvo (1983) pricing mechanism in firms' price-setting decision. So the two New Keynesian curves are derived using the optimality conditions of the private agents (households and firms). The derivation of these two curves for the standard New Keynesian model setup can be found, e.g., in Walsh (2003). The description of the New Keynesian model can also be found in Woodford (1996, 2003) and in Christiano, Eichenbaum, Evans (2001).

In solving their optimization problems, private agents are assumed to take the interest rate (showing up in the IS curve equation) as given. The interest rate, in turn, is set by the policy –maker — the central bank. In various studies of monetary policy issues (in the New Keynesian framework), it is normally assumed that the policy–maker uses some linear feedback rule to set the interest rate. In general, a feedback rule that is derived from the loss function minimization problem determines how the interest rate reacts to the expected values of the model's endogenous variables (inflation and output gap in the New Keynesian model) and the model's exogenous variables (various shocks, e.g., technology shock, preference shock, cost-push shock). Instrument rules, like Taylor–type rules, are designed to respond to the target variables (e.g., inflation and output gap). As is noted in the introduction, Taylor–type rules will be considered in a separate study.

Plugging the feedback rule into the IS curve equation, we obtain the model reduced form. Using the same New Keynesian equations (IS and Phillips curves), we can obtain different educed forms for different policy rules, i.e. other things being equal, the reduced forms structure depends on the policy rule. It depends not only on the type of it (Taylor or optimization–based), but, as is demonstrated by Evans and Honkapohja (2003b), on the assumption of the central bank about private agents expectations, resulting either in the fundamentals–based or in the expectations–based category of feedback rules.

After plugging some monetary policy rule of the central bank i_t, (we will talk about the types of optimal monetary policy rules later) the model can be written in the reduced form that has a general representation of a bivariate system with stationary AR(1) observable shocks process.

For derivations of our stability results we will allow for some generalization (as it is just a matter of notation compared to the bivariate model) and will consider

multivariate (not just bivariate) system with stationary AR(1) observable shocks process.

$$y_t = \alpha + A\hat{E}_t y_{t+1} + B w_t,$$

(In our bivariate case, $y_t = \begin{bmatrix} \pi_t & x_t \end{bmatrix}'$). The expectations operator with "hat" means that, in general, expectations of the private agents can be non-rational, which makes it possible to introduce adaptive learning in this model.

In our notation, the reduced form is written in such a way that it includes all factors that appear in the structural form. This means that the absence of some factor in the reduced form in our notation is expressed by the corresponding zero column of matrix B. (Note: We adopt such a notation here in order to be able later to consider different specifications of learning algorithms that include factors from different sets[4].) So our notation is the most general that can be.

In adaptive learning models of bounded rationality it is assumed that agents do not know rational expectations equilibrium and instead have their own understanding about the relation of variables in the model. The coefficients in this relation (that are called beliefs) are updated each period as new information on observed variables arrives (in this respect agents are modeled as if they are statisticians, or econometricians) For the beginning, we will assume that agents have the following perceived relation among the variables in the economy, which is called perceived law of motion (PLM)

$$y_t = a + \Gamma w_t \text{ , } (a = \begin{bmatrix} a_1 & a_2 \end{bmatrix}', \Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1k} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2k} \end{bmatrix} \text{ in the bivariate case})$$

that includes all components of w_t, a bit later we weaken this assumption.

Definition of RE, T-mapping

The rational expectations (RE) equilibrium is defined formally, in dynamic stochastic models as $E_t y_{t+1} = \hat{E}_t y_{t+1}$. (see, e.g., Sargent (1993) or Evans and Honkapohja (2001) for the meaning of the RE concept.) If we define the T–map as mapping of beliefs from PLM to the resulting coefficients (that are calculated after plugging PLM into actual dynamics (AD) of the economy) in actual law of motion (ALM),

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[4]An example of when a model reduced form may not include all shocks that are present as factors in the model structural form can be found in Evans and Honkapohja (2003), who used the New Keynesian model setup of Clarida, Gali, Gertler (1999).

$$T(a, \Gamma) = \left(Aa + \alpha, \ A\Gamma \begin{bmatrix} \rho_1 & & \\ & \ddots & \\ & & \rho_k \end{bmatrix} + B \right),$$

we will be able to write RE equilibrium condition as $T(a, \Gamma) = (a, \Gamma)$.

Now we will widen the set of PLMs considered. Let us start with the following definition.

Definition 1 *Active factors set is a subset of set of histories of w_{i_t} up to time t and a constant.*

Note that by active factors set we mean not those variables that agents are actually aware of at time t, but essentially those that are used by agents in their PLMs (the subset that may be smaller than the subset of actually available variables).

Agents now are assumed to have the perceived law of motion (PLM)

$y_t = a + \widetilde{\Gamma} \widetilde{w}_t$, (In the bivariate case, it looks as $a = \begin{bmatrix} a_1 & a_2 \end{bmatrix}', \widetilde{\Gamma} = \begin{bmatrix} \widetilde{\gamma}_{11} & \widetilde{\gamma}_{12} & \cdots & \widetilde{\gamma}_{1k'} \\ \widetilde{\gamma}_{21} & \widetilde{\gamma}_{22} & \cdots & \widetilde{\gamma}_{2k'} \end{bmatrix})$,

where $\widetilde{w}_t$ consists of the components of w_t included in agents' active factors set.

We renumber regressors that are included into agents' active factors set from 1 to k', the same is done with ρ's, and $\widetilde{B}$ is the corresponding part of matrix B from the reduced form. We will denote the set of subscripts taken from $\{1, ..., k\}$ corresponding to the active factors set as $\widetilde{I}$.

T-map: $\widetilde{T}(a, \widetilde{\Gamma}) = \left(Aa + \alpha, \ A\widetilde{\Gamma} \begin{bmatrix} \widetilde{\rho}_1 & & \\ & \ddots & \\ & & \widetilde{\rho}_{k'} \end{bmatrix} + \widetilde{B} \right).$

By analogy, one may try to write the REE condition as $\widetilde{T}(a, \widetilde{\Gamma}) = (a, \widetilde{\Gamma})$. However, in this case, it is clear that for the existence of a REE, agents have to include into their active factors set those factors w_{i_t} that correspond to non-zero columns of matrix B in the reduced form (PLM which consists only of such factors is a PLM that corresponds to the so called minimal state variable (MSV) solution). Also, in the above PLMs we have used the following assumption.

Assumption A *Agents include in their PLM of each endogenous variable all factors from their active factors set.*

(So we exclude situations when agents do not include into the PLM equation of one endogenous variable some factor having zero coefficient in matrix B of the reduced form, while including the same factor in the PLM equation of another

endogenous variable, with this factor having non-zero coefficient in matrix B of the reduced form. We assume that agents do not know the true structure of reduced form and use all the available info to form their expectations. So, if one factor is present in one PLM equation, it is present in other PLM equation. Essentially, Assumption A postulates that we write each agent's PLM equations in matrix form, without a priori setting coefficients in some factors to zero. In addition, we assume that all agents use the same set of factors (which in the matrix form means that they use the same vector). We also note here that a similar assumption on the matrix formulation of PLMs has been made by Giannitsarou (2003) and Honkapohja and Mitra (2006).)

Next propositions state necessary and sufficient conditions for existence and for uniqueness of RE equilibria in general multivariate model with stationary AR(1) observable shocks. These conditions are well-known, but we prefer to state them here for reader's convenience. To formulate following propositions, we return back to the initial numbering of shocks, denote constant in active factors set of agent as w_0 and take $\rho_0 = 1, B^0 = \alpha$. So, now i takes natural values from 0 to k. We will denote this set as I_0 and corresponding set of subscripts taken from $I_0 = \{0, 1, ..., k\}$ as $\widetilde{I}_0$. Note that constant is always included as factor in any active factors set, therefore 0 always belongs to I_0.

Proposition 2 *(Necessary and sufficient conditions for existence of a REE):* *Under assumption A, RE solution exists if and only if agents active factors set includes among others all w_i such that $B^i \neq \mathbf{0}$ in reduced form and $rank(\rho_i A - I) = rank(\rho_i A - I, B^i)$ for i such that $\det(\rho_i A - I) = 0$ and $B^i \neq \mathbf{0}$.*

Proof. see Appendix. ∎

Proposition 3 *(Necessary and sufficient conditions for existence and uniqueness of a REE):* *Under Assumption A, RE solution exists and unique if and only if agents active factors set includes among others all w_i such that $B^i \neq \mathbf{0}$ in reduced form and for **all** included w_i, $\det(\rho_i A - I) \neq 0$.*

Proof. see Appendix. ∎

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So, in what follows we always assume that Assumption A and the necessary and sufficient conditions[5] for existence of a REE hold true. Basically, we will assume that agents use in both equations of their PLM as minimum all the regressors that appear in right hand side of reduced form, and that RE solution (either unique or multiple) exists under such PLM. That is, in principle, we consider all possible PLMs that include as minimum all regressors that appear in right hand side of the reduced form and satisfy necessary and sufficient conditions of existence of the REE , as stated in proposition above.

3 General stability results under heterogeneous learning

Deriving conditions for stability of the multivariate forward–looking economic model under mixed RLS/SG heterogeneous learning for any (possibly different) degrees of inertia of agents, we naturally employ the general framework and notation from Honkapohja and Mitra (2006), who were first to formulate general criterion for stability of the structurally heterogeneous economy under mixed RLS/SG heterogeneous learning.

Honkapohja and Mitra (2006) consider the class of linear structurally heterogeneous models with S types of agents with different forecasts presented by

$$y_t = \alpha + \sum_{i=1}^{S} A_i \hat{E}_t^i y_{t+1} + B w_t,$$
$$w_t = F w_{t-1} + v_t,$$

where y_t is $n \times 1$ vector of endogenous variables, w_t is $k \times 1$ vector of exogenous variables, v_t -white noise, $\hat{E}_t^i y_{t+1}$ are (in general, non-rational) expectations of the endogenous variable by agent i, $M_w = \lim_{t\to\infty} w_t w_t'$ - is positive definite, F is such that w_t follows stationary VAR.

PLM is presented by

$$y_t = a_{i,t} + \Gamma_{i,t} w_t.$$

Part of agents, $i = 1, ..., S_0$, is assumed to use RLS learning algorithm while

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[5]The propositions above have similar meaning to Proposition 1 of Honkapohja and Mitra (2006), again, the condition requires matrices participating in deriving RE values of beliefs to be invertible. The idea is to state that we are aware of the fact that there are cases when an REE may not exist and of the conditions that are required for its existence (and uniqness).

others, $j = S_0 + 1, ..., S$, are assumed to use SG learning algorithm[6]. Moreover, all of them are assumed to use possibly different degrees of responsiveness to the updating function that are presented by different degrees of inertia δ_i, constant coefficients before common for all agents decreasing gain sequence in learning algorithm. (Honkapohja and Mitra (2006) use more general formulation of degrees of inertia.)

It is clear that such general formulation includes structurally homogeneous case of forward–looking model that we are interested in, $A_i = \sum_{i=1}^{S} \varsigma_i A$, where $\sum_{i=1}^{S} \varsigma_i = 1$, $1 > \varsigma_i > 0$ is weight of agents of type i. Really,

$$y_t = \alpha + \sum_{i=1}^{S} A_i \hat{E}_t^i y_{t+1} + B w_t = \alpha + \sum_{i=1}^{S} \varsigma_i A \hat{E}_t^i y_{t+1} + B w_t = \alpha + A \underbrace{\sum_{i=1}^{S} \varsigma_i \hat{E}_t^i y_{t+1}}_{\hat{E}_t^{aver} y_{t+1}} +$$

$B w_t$ is exactly formulation of the structurally homogeneous model, considered by Giannitsarou (2003).[7]

After denoting $z_t = (1, w_t)$ and $\Phi_{i,t} = (a_{i,t}, \Gamma_{i,t})$, the formal presentation of learning algorithms in this model can be written as follows.

RLS: for $i = 1, .., S_0$

$$\Phi_{i,t+1} = \Phi_{i,t} + \alpha_{i,t+1} R_{i,t}^{-1} z_t \left(y_t - \Phi_{i,t}' z_t \right)'$$

$$R_{i,t+1} = R_{i,t} + \alpha_{i,t+1} \left(z_{t-1} z_{t-1}' - R_{i,t} \right)$$

SG: for $j = S_0 + 1, ..., S$

$$\Phi_{j,t+1} = \Phi_{j,t} + \alpha_{j,t+1} z_t \left(y_t - \Phi_{j,t}' z_t \right)'$$

Honkapohja and Mitra (2006) that stability of REE, Φ_t, in this model is determined by stability of the ODE[8]:

$$\frac{d\Phi i}{d\tau} = \delta_i \left(T(\Phi')' - \Phi_i \right), \, i = 1, ..., S_0$$

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[6]Essentially, a part of agents are assumed to be more sophisticated in their learning (those that use RLS), because from the econometric point of view, RLS algorithm is more efficient because it uses information on the second moments.

[7]Heterogeneous learning in structurall homogeneous case was considered by Giannitsarou (2003) for more general class od self-referential linear stochastic models, that includes in itself the case of forward–looking models. Since our setup does not assume lagged endogenous variables, we are concentrating on structurally homogeneous case of forward–looking models that are a subclass of models considered by Giannitsarou and at the same time are a special case of the setup of Honkapohja and Mitra (2006).

[8]In general case, to obtain the associated ODE, one has to take math expectation of the RHS term (at the gain sequence) from the stochastic recursive algorithm (SRA) specification of a learning algorithm, with respect to the limiting distribution of the state vector. See Ch. 6.2 in Evans and Honkapohja (2001) for assumptions on the learning rule and state dynamics that have to hold so that we be able to apply the theory on SRA and local convergence analysis and the general formula for ODE (6.5) on p. 126.

$\frac{d\Phi j}{d\tau} = \delta_j M_z \left(T(\Phi')' - \Phi_j \right), \, j = S_0 + 1, ..., S,$

where $M_z = \lim_{t \to \infty} E z_t z_t'$.

The conditions of stability of this ODE give the general criterion of stability result for this class of models presented in Proposition 5 in Honkapohja and Mitra (2006).

In the economy above mixed RLS/SG learning converges globally (almost surely) to the minimal state variable (MSV) solution if and only if the matrices $D_1 \Omega$ and $D_w \Omega_F$ have eigenvalues with negative real parts, where

$$D_1 = \begin{pmatrix} \delta_1 I_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \delta_S I_n \end{pmatrix}, \Omega = \begin{pmatrix} A_1 - I_n & \cdots & A_S \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_S - I_n \end{pmatrix}$$

$$D_w = \begin{pmatrix} D_{w1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{wS} \end{pmatrix}, \Omega_F = \begin{pmatrix} F' \otimes A_1 - I_{nk} & \cdots & F' \otimes A_S \\ \vdots & \ddots & \vdots \\ F' \otimes A_1 & \cdots & F' \otimes A_S - I_{nk} \end{pmatrix},$$

$D_{wi} = \delta_i I_{nk}, i = 1, ..., S_0, \, D_{wj} = \delta_j \left(M_w \otimes I_n \right), j = S_0 + 1, ..., S.$

Since in our setup we assume the "diagonal" environment, namely $F = diag(\rho_1, ..., \rho_k)$, $M_w = diag\left(\frac{\sigma_1^2}{1-\rho_1^2}, ..., \frac{\sigma_k^2}{1-\rho_k^2} \right)$, the problem of finding stability conditions of both $D_1 \Omega$ and $D_w \Omega_F$ is simplified to finding stability conditions of $D_1 \Omega$ and $D_1 \Omega_{\rho_i}$, where Ω_{ρ_i} is obtained from Ω by substituting all A_h with $\rho_i A_h$, where $|\rho_i| < 1$ as w_t follows stationary VAR(1) process by setup of the model.

In our paper Bogomolova and Kolyuzhnov (2006) we used special blocked—diagonal structure of the matrix D_1 which is the feature of the dynamic environment in this class of models.(In a sense these positive diagonal D–matrices now may be called positive blocked—diagonal δ–matrices.), that allowed us to formulate the concept of δ–stability by analogy to the terminology of the concept of D–stability, studied for example in Johnson (1974).

Definition 4 *Given n, the number of endogenous variables, and S, the number of agent types, δ–stability is defined as stability of the economy under structurally heterogeneous mixed RLS/SG learning for any (possibly different) degrees of inertia of agents, $\delta > 0$.*

$\delta-$stability, thus formulated, has the same meaning in models with heterogeneous learning described above as has $E-$stability condition in models with homogeneous

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RLS learning. $E-$stability condition is the condition for asymptotic stability of an REE under homogeneous RLS learning. The REE of the model is stable if it is locally asymptotically stable under the following ODE:

$\frac{d\theta}{d\tau} = T(\theta) - \theta,$

where θ are the estimated parameters from agents PLMs, $T(\theta)$ is a mapping of PLM parameters into parameters of actual law of motion (ALM), which is obtained when we plug forecasts functions based on agents PLMs into the reduced form of the model and τ is a "notional" ("artificial") time. The fixed point of this ODE is the REE of the model.[9]

In case of structurally homogeneous learning, that we employ in current setup, the criterion simplifies to the Jacobians considered by Giannitsarou (2003) where she separated three types of learning. First to get structurally homogeneous learning as discussed before one has to replace, A_i with $\zeta_i A$.

The first type of heterogeneous learning is characterized by different initial perceptions of agents and equal degrees of inertia, this is so called transiently heterogeneous learning, by Honkapohja and Mitra (2006). The condition for stability under this learning is easily derived from the criterion above by setting all deltas to be equal, and setting S_0 to S or to 0 in order to get transiently heterogeneous RLS or SG learning respectively.

The second type of heterogeneous learning considered by Giannitsarou (2003)

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[9]Notice, that δ–stability conditions on the Jacobian in general forward–looking model of Honkapohja and Mitra (2006) do not depend on the particular equilibrium point (in case of multiple equilibria), because the system of differential equations is linear in this setup, in which case the first derivatives of the RHS of the associated ODE do not depend on a particular value of a RE equilibrium. So if stability conditions are satisfied for a given Jacobian, then all equilibrium points are stable. Convergence to a particular point depends on the initial conditions. In this paper we do not consider how equilibrium selection is done.

There may be specifications of models, like the ones considered in Giannitsarou (2003) or Evans and Honkapohja (2001), that include lags of endogenous variables, such that there may be multiple RE equilibria, as well as non-linear ODE system, implying that first derivatives will depend on the value of a particular steady state point and hence different Jacobians and different stability conditions. In case of different stability conditions, we may have situations when only one REE is learnable (the one for which the Jacobian satisfies the corresponding stability conditions) or some (or all) of the REE are learnable (if all Jacobians satisfy the corresponding stability conditions). As has been said above, the selection of a particular REE of those that are learnable depends on the initial conditions. There also may be cases when there will be switching from one learnable equilibrium to another, e.g. in Kasa (2004) for the case of constant-gain formulation of learning under which escapes from the area of attraction of one equilibrium point are more probable. The equilibrium selection issues require a separate study and are not considered here.

is learning when agents use different degrees of inertia and same type of learning algorithm (RLS or SG) (This is what Honkapohja and Mitra (2006) call a persistently heterogeneous learning in mild form). The Jacobians for this case are easily derived by setting S_0 to S or to 0 in order to get heterogeneous RLS or SG learning respectively and allowing for possibly different deltas.

The third type of heterogeneous learning considered by Giannitsarou (2003) is learning when different agent use different learning algorithms. It is characterized by possibly different initial perceptions, possibly different degrees of inertia, and different agents using different learning algorithms (RLS or SG). Such kind of learning Honkapohja and Mitra (2006) called persistently heterogeneous learning (in strong form).

The stability Jacobians for this case are derived by writing the general criterion of stability for structurally homogeneous case, i.e. by setting $A_i = \zeta_i A$, the thing we did from the very beginning.

Relation between the formulation of types of heterogeneous learning by Giannitsarou (2003) and Honkapohja and Mitra (2006) from the discussion above can be conveniently summarized in the following table[10]:

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[10]Note, that there is one type of heterogeneous learning that was not introduced by Giannitsarou (2003) and introduced here. It is heterogeneity in degrees of inertia when all types of agents use SG learning algorithm. Although Honkapohja and Mitra (2006) have general criterion for stability in this case (as discussed above), their formulation includes only forward-looking models. In general setup of self-referential structurally homogeneous models of Giannitsarou (2003) the stability conditions under such type of learning (in Giannitsarou (2003) notation, naturally extended from her proofs) would depend on the stability of the matrix $J_2^{SG}(\Phi_f) = diag(\delta_1,...,\delta_S) \otimes I \otimes M(\Phi_f) \bullet J_1^{LS}(\Phi_f)$, where Φ_f is an REE, $M(\Phi_f)$ defined analogously to M_z and $J_1^{LS}(\Phi_f)$ is Jacobian that defines stability in case of the first type of heterogeneity (different initial perceptions of agents) when all agents use RLS learning. For details, see Giannitsarou (2003). Again, it is clear that in forward–looking case these conditions of stability will fall under the general stability criterion of Honkapohja and Mitra (2006) with $S_0 = 0$ (see the table above).

Type of heterogeneity in learning	Type of learning algorithm	Assumptions in general H&M (2006) model	
		structurally heterogeneous	structurally homogeneous
		$A_i = \zeta_i \hat{A}_i$	$A_i = \zeta_i A$
I Different initial perceptions (transiently heterogeneous learning)	RLS SG	$\delta_i = \delta$ for all i, $S_0 = S$ $\delta_i = \delta$ for all i, $S_0 = 0$	
II Different degrees of inertia (persistently heterogeneous learning in mild form)	RLS SG	$S_0 = S$ $S_0 = 0$	
III Different learning algorithms (persistently heterogeneous learning in strong form)	RLS and SG		

One more comment regarding applicability of Honkapohja and Mitra (2006) constructions for analysis of our setup. Agents in setup of Honkapohja and Mitra (2006) are assumed to use PLM that corresponds to the so-called MSV solution, i.e. includes all factors that appear in the right hand side of the reduced form. However, Honkapohja and Mitra (2006) do not have restrictions in their proof of conditions for stability of the system on the matrix B. This means that we may, in principle, consider additional factors in learning that enter into reduced form with zero coefficients in matrix B for all agents. This means that we have to consider the criterion conditions for all possible PLMs that include (among others) all factors that appear in right hand side of the reduced form, satisfying conditions of existence and uniqueness specified in the previous chapter. And for this we have only to consider different matrices F, corresponding to different sets of factors in PLM. However, since all matrices F have special diagonal structure $F = diag(\rho_1, .., \rho_{\tilde{k}})$, as said above, the problem of analysis of $\delta-$stability is simplified to finding stability conditions of $D_1\Omega$ and $D_1\Omega_{\rho_i}$,where Ω_{ρ_i} is obtained from Ω by substituting all A_h with $\rho_i A_h$.where$i$ is taken from the corresponding set of factors included into PLM. And since $|\rho_i| < 1$ as w_t follows stationary VAR(1) process by setup of the model, it significantly simplifies the so-called aggregated economy conditions (necessary, and sufficient) for $\delta-$stability, allowing to state them without ρ' at all. (For details, please see Bogomolova and Kolyuzhnov (2006)).

4 Necessary, and sufficient conditions for δ–stability for structurally homogeneous models

First, we provide the reader with a set of sufficient conditions for δ–stability applicable to our setup. We will present the so–called "aggregated economy" sufficient condition for the case of structurally homogeneous model that was derived in Bogomolova and Kolyuzhnov (2006) from more general set of "aggregated economy" sufficient conditions for structurally heterogeneous models, and "equal sign" sufficient condition for the case of structurally heterogeneous bivariate economy that again was derived in Bogomolova and Kolyuzhnov (2006) using alternative definition of D–stability described in Johnson (1974) and subsequent alternative definition of δ–stability derived in Bogomolova and Kolyuzhnov (2006).

Proposition 5 *For structurally homogeneous economy: $A_h = \zeta_h A$, $\zeta_h > 0, \sum_{h=1}^{S} \zeta_h = 1$,to be δ–stable it is sufficient that at least one of the following limiting aggregated β–coefficients (that are coefficients before the expectation term of one–dimensional forward–looking aggregated economy model. For details see Bogomolova and Kolyuzhnov (2006)): $\max_i \sum_j |a_{ij}|$ and $\max_j \sum_i |a_{ij}|$ be less than one.*

Proof. see Bogomolova and Kolyuzhnov (2006) ∎

Proposition 6 *In case $n = 2$, structurally heterogeneous economy is $\delta-$stable if Ω is stable and an "equal sign" condition holds true, where the "equal sign" condition looks in general case as follows*

$\det(-\rho_l A_i) \geq 0$, $[\det mix(-\rho_l A_i, -\rho_l A_j) + \det mix(-\rho_l A_j, -\rho_l A_i)] \geq 0$, $i \neq j$, $M_1(-\rho_l A_i) \geq 0$,

or

$\det(-\rho_l A_i) \leq 0$, $[\det mix(-\rho_l A_i, -\rho_l A_j) + \det mix(-\rho_l A_j, -\rho_l A_i)] \leq 0$, $i \neq j$, $M_1(-\rho_l A_i) \leq 0$ for all $l = 0, 1, ..., k$ $(\rho_0 = 1)$

(Here we introduce the concept of a pairwise mixed economy which is an economy characterized by a matrix of structural parameters composed by mixing columns of a pair of matrices $\rho_l A_i$, $\rho_l A_j$, for any $i, j = 1, ..., S$.)

Proof. see Bogomolova and Kolyuzhnov (2006) ∎

It is also possible to derive some necessary, and sufficient conditions of δ–stability in structurally homogeneous case in terms of the values of eigenvalues of matrix of structural parameters of the reduced form, A. It is possible by direct application of so-called characteristic equation approach, when one requires that all the roots of the polynomial (that are eigenvalues of Jacobian matrix) are less than zero for stability, the latter being equivalent to well-known Routh–Hurwitz conditions.

Proposition 7 *If all eigenvalues of A are real and less than one, then the structurally homogeneous system with two agents is $\delta-$stable (that is stable under three types of heterogeneous learning: agents with different initial perceptions with RLS or SG learning, agents with possibly different degrees of inertia with RLS or SG learning, and agents with different learning algorithms, RLS and SG). For the structurally homogeneous system with any number of agents to be $\delta-$stable, it is necessary that all real roots of A are less than one. (This gives a test for non–δ–stability)*

Proof. see Appendix ∎

In proof of the proposition above we, using the structure of the Jacobian matrices in our setup, have derived a sufficient condition for stability under all three types of heterogeneous learning with two agent types. We did this using the criterion for stability of Honkapohja and Mitra (2006). For the case of real roots of A, we have shown that in this setup, analysis of stability of a particular Jacobian turns into the analysis of stability of A, which gives us very simple eigenvalues conditions. Also, using the general criterion of Honkapohja and Mitra (2006), we have proved here necessary conditions for δ–stability (the failure of which is sufficient for non–δ–stability) for the case of arbitrary number of agent types.

5 Optimal policy rules and structure of reduced forms

Here we describe the types of optimal policy rules that are analyzed in this study. The policy-maker is assumed to use the loss function minimization problem, which comes from the flexible inflation targeting approach (a policy regime adopted in several countries in the 1990s), described and defended by Svensson (1999). The

central bank here has two options: adopt a discretionary policy, by solving the problem every period, or commit to the rule which is once and for all derived from the minimization of the infinite horizon loss function. Svensson (1999) and Cecchetti (2001) advocate the first option, which is essentially commitment to a certain behavior (minimizing the loss function) with reconsidering the optimal rule every period (so that to take into account new information). They provide various arguments, like inefficiency (in general) of instrument rules designed to respond only to target variables, or how monetary policy decisions are made in practice.

The infinite horizon loss function of the policy maker for flexible inflation targeting approach looks as follows.

$$\frac{1}{2} E_t \sum_{i=0}^{\infty} \beta^i \left[\alpha \left(x_{t+i} - \bar{x} \right)^2 + \left(\pi_{t+i} - \bar{\pi} \right)^2 \right]$$

Since we assume discretionary policy of the policy maker according to the discussion above, the problem of minimizing loss function simplifies to solving each period

$$\frac{1}{2} \left[\alpha \left(x_t - \bar{x} \right)^2 + \left(\pi_t - \bar{\pi} \right)^2 \right] + F_t$$

subject to

$\pi_t = c_2 + \lambda x_t + F_t$ (government takes the remainder terms of the loss function F_t, and constraint, $F_t = \beta \widehat{E}_t \pi_{t+1} + \chi_2 w_t$, as given).

The classification below of the loss–function–optimization–based rules into fundamentals based and expectations based is due to Evans and Honkapohja (2003b). Derivations of theses rules and corresponding reduced forms were done by Evans and Honkapohja (2003b) for a slightly more narrow setup than we assume here (we assume general structure of autoregressive shocks), therefore we basically repeat their steps in derivations that follow below, extending them for our setup.

5.1 Expectations–based optimal policy rules

Expectations–based policy rule implies the central bank's reaction to (possibly non-rational) expectations of private agents, assuming that these expectations are observable (or can be estimated). Its general form is $i_t = \delta_0 + \delta_\pi \hat{E}_t \pi_{t+1} + \delta_x \hat{E}_t x_{t+1} + \delta'_w w_t$. The coefficients of this rule are obtained by solving the equilibrium conditions

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(structural equations with non–rational expectations of private agents and FOC of the central bank).

Thus, the optimal expectations based optimal policy rule is derived from F.O.C.

$$\lambda \left(\pi_t - \bar{\pi} \right) + \alpha \left(x_t - \bar{x} \right) = 0$$

and

$$x_t = c_1 - \phi \left(i_t - \widehat{E}_t \pi_{t+1} \right) + \widehat{E}_t x_{t+1} + \chi_1' w_t$$
$$\pi_t = c_2 + \lambda x_t + \beta \widehat{E}_t \pi_{t+1} + \chi_2' w_t$$

From this we get expectations based policy rule:

$i_t = \delta_0 + \delta_\pi \hat{E}_t \pi_{t+1} + \delta_x \hat{E}_t x_{t+1} + \delta_w' w_t$, where

$$\delta_0 = - \left(\lambda^2 + \alpha \right)^{-1} \varphi^{-1} \left(\lambda \bar{\pi} + \alpha \bar{x} - \lambda c_2 - \left(\alpha + \lambda^2 \right) c_1 \right)$$
$$\delta_\pi = 1 + \left(\lambda^2 + \alpha \right)^{-1} \varphi^{-1} \lambda \beta$$
$$\delta_x = \varphi^{-1}$$
$$\delta_w = \varphi^{-1} \chi_1 + \left(\lambda^2 + \alpha \right)^{-1} \varphi^{-1} \lambda \chi_2$$

After plugging this policy rule into the IS curve equation, we get the following reduced form.

$$y_t = c^E + A^E \hat{E}_t y_{t+1} + \chi^E w_t,$$
$$w_t = F w_{t-1} + \nu_t,$$
$$y_t = \begin{bmatrix} \pi_t & x_t \end{bmatrix}',$$

where $A^E = \begin{pmatrix} \beta \alpha \left(\lambda^2 + \alpha \right)^{-1} & 0 \\ -\beta \lambda \left(\lambda^2 + \alpha \right)^{-1} & 0 \end{pmatrix}$

$$F = diag(\rho_i)$$
$$|\rho_i| < 1, \ \nu_{i_t} \sim iid \left(0, \sigma_i^2 \right), \ i = 1, ..., n$$
$$c^E = \begin{pmatrix} c_2 + \lambda \left(c_1 - \varphi \delta_0 \right) & c_1 - \varphi \delta_0 \end{pmatrix}'$$
$$\chi^E = \begin{pmatrix} \chi_2' \left[1 - \frac{\lambda^2}{\lambda^2 + \alpha} \right] \\ -\frac{\lambda^2}{\lambda^2 + \alpha} \chi_2' \end{pmatrix}$$

Note that REE solution is not needed neither for deriving A^E matrix, nor for deriving coefficients of optimal expectations–based policy rule. REE solution will be needed for deriving optimal fundamentals–based policy rule, and therefore will be derived in the corresponding part of the text.

5.2 Fundamentals based optimal policy rules

In general fundamentals based policy rule (not necessarily optimal) has the form

$$i_t = \psi_0 + \sum_{i=1}^{n} \psi_{w_i} w_{i_t} = \psi_0 + \psi'_w w_t$$

We will show later that there exists unique fundamentals based optimal policy rule in this setup and will find it.

Plugging this policy rule into the structural form

$$x_t = c_1 - \phi \left(i_t - \widehat{E}_t \pi_{t+1} \right) + \widehat{E}_t x_{t+1} + \chi'_1 w_t$$

$$\pi_t = c_2 + \lambda x_t + \beta \widehat{E}_t \pi_{t+1} + \chi'_2 w_t$$

we get the reduced form

$$y_t = c^F + A^F \widehat{E}_t y_{t+1} + \chi^F w_t,$$

$$w_t = F w_{t-1} + \nu_t,$$

$$y_t = \begin{bmatrix} \pi_t & x_t \end{bmatrix}',$$

where $A^F = \begin{pmatrix} \beta + \lambda \varphi & \lambda \\ \varphi & 1 \end{pmatrix}$

$$F = diag(\rho_i)$$

$$|\rho_i| < 1, \ \nu_{i_t} \sim iid\left(0, \sigma_i^2\right), \ i = 1, ..., n$$

$$c^F = \begin{pmatrix} c_1 - \varphi\psi_0 & c_2 + \lambda\left(c_1 - \varphi\psi_0\right) \end{pmatrix}'$$

$$\chi_F = \begin{pmatrix} \lambda\left(-\varphi\psi'_w + \chi'_1\right) + \chi'_2 \\ -\varphi\psi'_w + \chi'_1 \end{pmatrix}$$

Optimal fundamentals–based rule, under the central banks' discretionary policy, is obtained from the loss function minimization, with the central bank assuming that private agents have RE. Its general form, when the REE structure is $y_t = a + \Gamma w_t$, is $i_t = \psi_0 + \psi'_w w_t$, where w_t is a vector of exogenous variables. Using the equilibrium conditions (economy's structural equations with REE structure entering them and the FOC of the central bank's optimization problem), we obtain coefficients of the REE and of the optimal fundamentals based policy rule.

To get REE, one has to write down ALM using Phillips curve $\pi_t = c_2 + \lambda x_t + \beta \widehat{E}_t \pi_{t+1} + \chi'_2 w_t$, FOC of the government optimization problem $\lambda\left(\pi_t - \bar{\pi}\right) + \alpha\left(x_t - \bar{x}\right) = 0$ and PLM in general form, $y_t = a + \Gamma w_t$, and then according to RE principle equate coefficients of resulting ALM (T–mapping) with corresponding coefficients of PLM.

Resulting ALM looks as

$$\pi_t = \frac{c_2 + \lambda[\lambda\bar{\pi} + \alpha\bar{x}]}{\lambda^2 + \alpha} + \frac{\alpha\beta}{\lambda^2 + \alpha}\left[a_1 + \gamma_{11}\rho_1 w_{1t} + ... + \gamma_{1n}\rho_n w_{nt}\right] + \frac{\alpha}{\lambda^2 + \alpha}\chi'_2 w_t,$$

$$x_t = \frac{\lambda\bar{\pi} + \alpha\bar{x}}{\alpha} - \frac{\lambda}{\alpha}\pi_t$$

REE looks as

$$\pi_t = a_1^* + \sum_{i=1}^{n} \gamma_{1i}^* w_{it}$$

$$x_t = a_2^* + \sum_{i=1}^{n} \gamma_{2i}^* w_{it},$$

where

$a_1^* = \frac{c_2 + \lambda[\lambda\bar{\pi} + \alpha\bar{x}]}{\lambda^2 + \alpha(1-\beta)}$, $a_2^* = \frac{\lambda\bar{\pi} + \alpha\bar{x}}{\alpha} - \frac{\lambda}{\alpha}a_1^* = \frac{-\frac{\lambda}{\alpha}c_2 + (1-\beta)[\lambda\bar{\pi} + \alpha\bar{x}]}{\lambda^2 + \alpha(1-\beta)}$, $\gamma_{1i}^* = \frac{\alpha\chi_{2i}\rho_i}{\alpha(1-\beta\rho_i) + \lambda^2}$, $\gamma_{2i}^* = -\frac{\lambda}{\alpha}\gamma_{1i}^* = -\frac{\lambda\chi_{2i}\rho_i}{\alpha(1-\beta\rho_i) + \lambda^2}$, $i = 1, ..., n$

To get optimal fundamentals based policy rule, one has to express i_t using IS curve, $x_t = c_1 - \phi\left(i_t - \widehat{E}_t\pi_{t+1}\right) + \widehat{E}_t x_{t+1} + \chi_1' w_t$, plugging in it RE solution derived above.

$$i_t = -\frac{1}{\varphi}\left(a_2^* + \sum_{i=1}^{n}\gamma_{2i}^* w_{it}\right) + \left(a_1^* + \sum_{i=1}^{n}\gamma_{1i}^*\rho_i w_{it}\right) + \frac{1}{\varphi}\left(a_2^* + \sum_{i=1}^{n}\gamma_{2i}^*\rho_i w_{it}\right) + \frac{1}{\varphi}\chi_1' w_t$$

As a result, optimal fundamentals based policy rule looks as

$i_t = \psi_0^* + \psi_w^{*\prime} w_t$, where

$\psi_0^* = a_1^*$

$\psi_w^* = \frac{1}{\varphi}\left[\left(\begin{array}{ccc}\gamma_{21}(\rho_1 - 1) & \dots & \gamma_{2n}(\rho_n - 1)\end{array}\right) + \chi_1\right] + \left(\begin{array}{ccc}\gamma_{11}\rho_1 & \dots & \gamma_{1n}\rho_n\end{array}\right)$.

In both cases, we plug the corresponding policy rule into the structural equations and obtain the corresponding reduced form of the model. These RFs were studied for stability under homogeneous RLS learning in the Clarida, Gali and Gertler (1999) formulation of the New Keynesian model by Evans and Honkapohja (2003b), who have derived the stability results for expectations based rule and instability results for fundamentals based rule. We study stability and instability for the two categories of rules under heterogeneous learning in a general setup of the structurally homogeneous New Keynesian model. When our model is structurally homogeneous and private agents are heterogeneous only in the way they form expectations (and learn), the two structural equations of the New Keynesian model will have, as the expectations term, weighted expectations of different agent types (with weights being the mass of each agent type), which will be multiplied by a common structural parameters matrix. The expectations–based policy rule, in this setup, will also depend on the weighted expectations of private agents, and so will the economy's reduced form.

6 Stability problem in the New Keynesian model

After deriving conditions for δ–stability for our class of models, we are ready to check for δ–stability of our New Keynesian model under different types of optimal monetary policy rules. For this we have to test the resulting (after substituting for particular type of policy rule) A matrix of the reduced form with respect to applicability of sufficient and necessary conditions of δ–stability.

Proposition 8 *General New Keynesian model with stationary AR(1) observable shocks process is δ–stable when optimal expectations-based policy rule is applied.*

Proof. We have that corresponding A matrix in optimal expectations-based policy rule case is $A^E = \begin{pmatrix} \beta\alpha\left(\lambda^2 + \alpha\right)^{-1} & 0 \\ -\beta\lambda\left(\lambda^2 + \alpha\right)^{-1} & 0 \end{pmatrix}$. Using the "equal sign" condition in proposition above, we have that Ω is stable, since its eigenvalues are determined from the following characteristic equation $\det\left(A^E - I_2\left(1 + \mu\right)\right)\left(1 + \mu\right)^{2(S-1)} = 0$ and therefore are equal to -1 and $\beta\alpha\left(\lambda^2 + \alpha\right)^{-1} - 1$, i. e. negative, and $\det\left(-\rho_l A_i\right) = 0$, $\left[\det mix\left(-\rho_l A_i, -\rho_l A_j\right) + \det mix\left(-\rho_l A_j, -\rho_l A_i\right)\right] = 0$, $i \neq j$, $M_1(-\rho_l A_i) = -\rho_l\zeta_h\beta\alpha\left(\lambda^2 + \alpha\right)^{-1} \geq (\leq) 0$,for all $l = 0, 1, ..., k$ $(\rho_0 = 1)$, so "equal sign" condition holds true. Notice, that using the "aggregated economy" sufficient condition from proposition, we can write down two aggregated β–coefficients in this case. These are $\beta_1^{\max} = \max_i \sum_j |a_{ij}| = \max\left\{\beta\alpha\left(\lambda^2 + \alpha\right)^{-1}, \beta\lambda\left(\lambda^2 + \alpha\right)^{-1}\right\}$ and $\beta_2^{\max} = \max_j \sum_i |a_{ij}| = \beta\left(\alpha + \lambda\right)\left(\lambda^2 + \alpha\right)^{-1}$. It is clear that both coefficients are less than one if $\lambda \geq 1$. So, the "aggregated economy" sufficient condition for δ–stability is more restrictive condition compared to "equal sign" condition since it requires additional assumptions on the structure of the economy. However, it can be with success applied in more than two dimensional economies where similar "equal sign" conditions are not sufficient for δ–stability (see Bogomolova and Kolyuzhnov (2006)). ∎

Note, that Evans and Honkapohja (2003b) have the same result for a homogeneous learning of partial case of this formulation.

Proposition 9 *General New Keynesian model with stationary AR(1) observable shocks process is non–δ–stable, when fundamentals–based policy rule (including optimal fundamentals-based policy rule) is applied.*

Proof. We have that corresponding A matrix in fundamentals–based policy rule case is $A^F = \begin{pmatrix} \beta + \lambda\varphi & \lambda \\ \varphi & 1 \end{pmatrix}$. Using the "eigenvalues" necessary condition from proposition[11], we get the eigenvalues of this matrix. $\mu_{1,2} = 1 + \frac{\beta+\lambda\varphi-1}{2} \pm \sqrt{\left(\frac{\beta+\lambda\varphi-1}{2}\right)^2 + \lambda\varphi}$. Both of these eigenvalues are real and eigenvalue $\mu_1 = 1 + \frac{\beta+\lambda\varphi-1}{2} + \sqrt{\left(\frac{\beta+\lambda\varphi-1}{2}\right)^2 + \lambda\varphi}$ is greater than one. So, the sufficient condition for non–δ–stability is satisfied. ∎

Again, Evans and Honkapohja (2003b) have the same result for a homogeneous learning of partial case of this formulation.

The first proposition means that RE in this model resulting after implementing optimal expectations-based policy rule, is stable under recursive least squares (RLS) and stochastic gradient (SG) homogeneous learning and three types of heterogeneous learning: agents with different initial perceptions with RLS or SG learning, agents with different degrees of inertia with RLS or SG learning, and agents with different learning algorithms, RLS and SG. The second proposition claims that the RE of the same model but with fundamentals–based policy rule is always unstable under any type of heterogeneous and homogeneous learning of agents.

7   Conclusion

We have used the environment of the New Keynesian model to explore the question of stability of two categories of monetary policy rules under the assumption of heterogeneous learning of agents.

These two categories were introduced by Evans and Honkapohja (2003b), and this division is based on the assumption about the central bank's perception of private agents' expectations: RE or possibly non-rational. Under the central bank assuming private agents to have RE, the fundamentals-based rule is obtained, while the case of the central bank assuming possibly non-rational expectations of private agents results in the fundamentals-based rule.

The purpose of this research was, on the one hand, to explore whether, given structural homogeneity of the model, heterogeneity in learning of agents influences

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[11]In principle, we could also use our necessary conditions for delta-stability (derived in our other paper) to show instability of the fundamentals-based rule. But this may be more difficult to check than the necessary conditions on eigenvalues that we derived in this paper. Besides, our eigenvalues necessary conditions work for the case of arbitrary number of agent types.

stability results implied by the application of either of the two categories of policy rules.

Using the general criterion for stability of Honkapohja and Mitra (2006) and our sufficient delta-stability conditions, for the case of heterogeneous learning, we obtain results similar to those obtained by Evans and Honkapohja (2003b) for the case of homogeneous learning. In particular, under the fundamentals-based policy rule, the model economy is always unstable, so there is no convergence to the associated REE of the model, while there is stability under the optimal expectations-based rule and the economy converges to the REE corresponding to optimal monetary policy without commitment.

The above-described results have been obtained using only the structure of the model, so there is no dependence on heterogeneity of any type considered. This implies that in the structurally homogeneous New Keynesian model, there is no dependence of stability results on heterogeneity of learning, so the representative agent hypothesis is applicable in this setup.

While checking the applicability of this hypothesis in structurally homogeneous New Keynesian model with heterogeneous learning of agents, we have not considered Taylor–type rules, which do not fall under the classification of Evans and Honkapohja (2003b).This issue will be considered in a separate study.

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A   Appendix

Proof of Propositions 2 and 3:

PLM in general form, $y_t = a + \Gamma w_t$. If w_i is not included in PLM, it is reflected in corresponding zero column of Γ. The REE conditions can be written as $(\rho_i A - I_n) \begin{bmatrix} \gamma_{1i} \\ \vdots \\ \gamma_{ni} \end{bmatrix} + B^i = 0$, $i \in I_0$.

It is clear, that in case i is not included into active factors set, that is $\begin{bmatrix} \gamma_{1i} \\ \vdots \\ \gamma_{ni} \end{bmatrix} = 0$,then in order to have REE solution, B^i has to be equal to 0, that is one can omit in PLM only those factors, that have zero column of B in the reduced form. Equivalently, it is clear that if $B_i \neq 0$, then in order to have REE solution, one should not have $\begin{bmatrix} \gamma_{1i} \\ \vdots \\ \gamma_{ni} \end{bmatrix} = 0$, that is have to include w_i into active factors set.

In case i is included in active factors set, that is $\begin{bmatrix} \gamma_{1i} \\ \vdots \\ \gamma_{ni} \end{bmatrix} \neq 0$, the REE solution exists if and only if the following condition holds true.

$B^i = 0$, or ($B^i \neq 0$ and $\det(\rho_i A - I) \neq 0$), or ($B^i \neq 0$ and $\det(\rho_i A - I) = 0$ and $rank(\rho_i A - I) = rank(\rho_i A - I, B^i)$).

Combining the two cases we get the statement in the proposition 2.

For the proposition 3, one has only to refine the last conditions to guarantee uniqueness of solution.

In case i is included in active factors set, that is $\begin{bmatrix} \gamma_{1i} \\ \vdots \\ \gamma_{ni} \end{bmatrix} \neq 0$, the REE solution exists and unique if and only if the following condition holds true.

$\det(\rho_i A - I) \neq 0$.

Proof of Proposition 6:

Necessary conditions and sufficient conditions in terms of eigenvalues for structurally homogeneous case.

$$\det\left(\Omega - D^{-1}\mu I\right) = \det \begin{bmatrix} A_1 - \left(1 + \frac{\mu}{\delta_1}\right)I & \cdots & A_S \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_S - \left(1 + \frac{\mu}{\delta_S}\right)I \end{bmatrix} = 0,$$

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where $A_i = \zeta_i \bar{A}$, $\sum \zeta_i = 1$

It is clear from the structure of the matrix above that $\mu = -\delta_{i_0}$ is a root if and only if at least one of the following holds true: $\bar{A}$ is singular or there exists at least one other δ_j that equals to δ_{i_0}.

(If $\bar{A}$ is singular than $\mu = \delta_i$, $i = 1, ..., S$ are the roots. That is if none of $-\delta$'s is the root than $\bar{A}$ is non-singular.)

Assume that A is non-singular and all δ_i's are different, that is that none of $-\delta$'s is the root.

If there are other roots than $-\delta_i$'s (case of eigenvalues $\mu = -\delta_i < 0$ isobvious) then they satisfy the characteristic equation for getting eigenvalues of $D\Omega$ that are not equal to $-\delta_i$:

$$\det \left[\frac{-A_1}{1 + \frac{\mu}{\delta_1}} + ... + \frac{-A_S}{1 + \frac{\mu}{\delta_S}} + I \right] = 0$$
$$A_i = \zeta_i \bar{A}, \sum \zeta_i = 1$$

Developing it further, one gets

$$\det \left[\bar{A} \left(\frac{-\zeta_1}{1 + \frac{\mu}{\delta_1}} + ... + \frac{-\zeta_S}{1 + \frac{\mu}{\delta_S}} \right) + I \right] = 0$$

and finally

$$\lambda_k \left(\frac{\zeta_1}{1 + \frac{\mu}{\delta_1}} + ... + \frac{\zeta_S}{1 + \frac{\mu}{\delta_S}} \right) = 1, \text{ for those eigenvalues of } \bar{A}, \lambda_k \text{ that do not equal to}$$

zero. If all $\lambda_k = 0$, then $\bar{A}$ is zero matrix and the only eigenvalues of $D\Omega$ are $-\delta_i$'s.

As complex eigenvalues of a real matrix $\bar{A}$ come in conjugate pairs, the systems above are equivalent to

$$\text{Re}\,(\lambda_k) \,\text{Re} \left(\frac{\zeta_1}{1 + \frac{\mu}{\delta_1}} + ... + \frac{\zeta_S}{1 + \frac{\mu}{\delta_S}} \right) - \text{Im}\,(\lambda_k) \,\text{Im} \left(\frac{\zeta_1}{1 + \frac{\mu}{\delta_1}} + ... + \frac{\zeta_S}{1 + \frac{\mu}{\delta_S}} \right) = 1$$

$$\text{Im}\,(\lambda_k) \,\text{Re} \left(\frac{\zeta_1}{1 + \frac{\mu}{\delta_1}} + ... + \frac{\zeta_S}{1 + \frac{\mu}{\delta_S}} \right) + \text{Re}\,(\lambda_k) \,\text{Im} \left(\frac{\zeta_1}{1 + \frac{\mu}{\delta_1}} + ... + \frac{\zeta_S}{1 + \frac{\mu}{\delta_S}} \right) = 0$$

for each pair of conjugate eigenvalues.

In case of real eigenvalue, $\text{Im}\,(\lambda_k) = 0$, the corresponding system simplifies to

$$\text{Re}\,(\lambda_k) \left(\frac{\zeta_1}{1 + \frac{\mu}{\delta_1}} + ... + \frac{\zeta_S}{1 + \frac{\mu}{\delta_S}} \right) = \lambda_k \left(\frac{\zeta_1}{1 + \frac{\mu}{\delta_1}} + ... + \frac{\zeta_S}{1 + \frac{\mu}{\delta_S}} \right) = 1$$

For any S we have that for eigenvalues μ to be negative it is necessary, that $\frac{\frac{1}{\lambda_k} - 1}{\frac{1}{\lambda_k \delta_1 ... \delta_S}} > 0$, therefore that $\lambda_k < 1$

For $S = 2$, the system corresponding to real eigenvalue looks as

$$\lambda_k \left(\frac{\zeta_1}{1+\frac{\mu}{\delta_1}} + \frac{\zeta_2}{1+\frac{\mu}{\delta_2}} \right) = 1$$

$$\mu^2 + \mu \frac{\frac{1}{\lambda_k}\left(\frac{1}{\delta_1}+\frac{1}{\delta_2}\right)-\left(\frac{\zeta_1}{\delta_2}+\frac{\zeta_2}{\delta_1}\right)}{\frac{1}{\lambda_k \delta_1 \delta_2}} + \frac{\frac{1}{\lambda_k}-1}{\frac{1}{\lambda_k \delta_1 \delta_2}} = 0$$

Routh–Hurwitz conditions for negativity of real parts of μ are necessary and sufficient and look as

$$\frac{\frac{1}{\lambda_k}-1}{\frac{1}{\lambda_k \delta_1 \delta_2}} > 0$$

$$\frac{\frac{1}{\lambda_k}\left(\frac{1}{\delta_1}+\frac{1}{\delta_2}\right)-\left(\frac{\zeta_1}{\delta_2}+\frac{\zeta_2}{\delta_1}\right)}{\frac{1}{\lambda_k \delta_1 \delta_2}} > 0$$

The system of inequalities above is equivalent to

$$\lambda_k < 1$$

$$\lambda_k < \frac{\frac{1}{\delta_1}+\frac{1}{\delta_2}}{\frac{\zeta_1}{\delta_2}+\frac{\zeta_2}{\delta_1}}.$$

And since $\frac{\frac{1}{\delta_1}+\frac{1}{\delta_2}}{\frac{\zeta_1}{\delta_2}+\frac{\zeta_2}{\delta_1}} > 1$, as $\frac{1-\zeta_1}{\delta_1} + \frac{1-\zeta_2}{\delta_2} > 0$, the last system of inequalities is equivalent to $\lambda_k < 1$.

Thus, we get sufficient condition for stability for case $S = 2$, that all eigenvalues of $\bar{A}$ are real and less than 1, and necessary condition for stability for any S is that all real eigenvalues of $\bar{A}$ have to be less than 1.